The LQG – String: Loop Quantum Gravity Quantization of String Theory

I. Flat Target Space

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Abstract

We combine I. background independent **Loop Quantum Gravity (LQG)** quantization techniques, II. the mathematically rigorous framework of **Algebraic Quantum Field Theory (AQFT)** and III. the theory of integrable systems resulting in the invariant **Pohlmeyer Charges** in order to set up the general representation theory (superselection theory) for the closed bosonic quantum string on flat target space.

While we do not solve the, expectedly, rich representation theory completely, we present a, to the best of our knowledge new, non – trivial solution to the representation problem. This solution exists 1. for any target space dimension, 2. for Minkowski signature of the target space, 3. without tachyons, 4. manifestly ghost – free (no negative norm states), 5. without fixing a worldsheet or target space gauge, 6. without (Virasoro) anomalies (zero central charge), 7. while preserving manifest target space Poincaré invariance and 8. without picking up UV divergences.

The existence of this stable solution is, on the one hand, exciting because it raises the hope that among all the solutions to the representation problem (including fermionic degrees of freedom) we find stable, phenomenologically acceptable ones in lower dimensional target spaces, possibly without supersymmetry, that are much simpler than the solutions that arise via compactification of the standard Fock representation of the string. On the other hand, if such solutions are found, then this would prove that neither a critical dimension (D=10,11,26) nor supersymmetry is a prediction of string theory. Rather, these would be features of the particular Fock representation of current string theory and hence would not be generic.

The solution presented in this paper exploits the flatness of the target space in several important ways. In a companion paper we treat the more complicated case of curved target spaces.

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Contents

1 Introduction

String Theory (ST) [1] and Loop Quantum Gravity (LQG) [2] (see [3] for recent reviews) are currently the two major approaches towards a quantum theory of gravity. They are complementary in many senses. For example, in ST a central idea is the *Unification of all Forces (UF)* while in LQG the unification of the *Background Independence (BI) Principle* with the principles of Quantum Theory is considered as the most important guideline. Hence, in ST the BI is currently not implemented and vice versa LQG presently does not put any constraints on the matter content of the world. It is not clear if any of these principles or both should be realized in quantum gravity at all, however, historically there is evidence for the success of both. On the one hand, the non-renormalizable Fermi model of the weak interaction was replaced by the renormalizable electroweak theory which unifies the weak and electromagnetic interaction. On the other hand the puzzles of non-relativistic quantum mechanics (e.g. negative energy particles) were resolved by unifying special relativity and quantum mechanics in QFT ("second quantization").

In the absence of experimental input (so far) it is therefore worthwhile to keep our minds open and push complementary ideas to their frontiers and to learn from the advantages and disadvantages of competing programmes. It is the purpose of this paper to make a small contribution to that extent. Namely, we ask the question:

Can the BI methods of LQG be employed in order to provide an alternative quantization of ST? If yes, what are the differences?

By ST we mean here old-fashioned perturbative string theory and not its (yet to be defined) M – Theory generalization.

In this paper we precisely define the general quantization problem for the closed, bosonic string on flat (Minkowski) target space. Namely, we set up the representation theory for the closed, bosonic string. We can fruitfully combine three different frameworks:

- I. *Background Independent LQG*
  The string can be viewed as a worldsheet diffeomorphism invariant, two – dimensional QFT. Hence it precisely falls into class of theories that can be quantized by LQG methods. Moreover, LQG provides a general framework for how to implement quantum constraints without gauge fixing.

- II. *Algebraic Quantum Field Theory (AQFT)*
  The Haag – Kastler approach to QFT [4] provides a clean distinction between the physical object to be quantized, namely the algebra of physical observables, and the corresponding set of representations (Hilbert spaces) thereof. The latter can be viewed as different phases of the the theory which may be or may be not realized in nature. AQFT provides very powerful tools in order to solve the classification problem of the corresponding representations, see e.g. [5] for a recent review. For infinite dimensional systems such as the string there is no Stone – von Neumann uniqueness theorem [6] and the corresponding representation theory is usually very complex.

- III. *Integrable Systems (Pohlmeyer Charges)*
  In constrained dynamical systems it is highly non – trivial to identify the gauge invariant observables of the theory. It is even harder to find faithful representations of the corresponding, usually highly non – linear, commutation relations. Fortunately, for the closed bosonic string on flat target space the complete set of gauge invariant observables, the **Pohlmeyer Charges**, has been found [7] and their classical Poisson algebra is under complete control [8, 9, 10]. This will come to the surprise of most string theorists who are used to work in the so – called conformal worldsheet and/or lightcone target space gauge. In contrast, these observables are manifestly gauge invariant and manifestly Lorentz covariant.

Hence, what we are looking for is a background independent, that is, gauge invariant (in the sense of **LQG**) and Lorentz invariant, representation (in the sense of **AQFT**) of the **Pohlmeyer Algebra**. Notice that by definition these representations do not allow us to pick either a worldsheet gauge (such as the conformal gauge) nor a target space gauge (such as the lightcone gauge). Consequently, the problem can be set up directly for the Nambu – Goto String rather than the Polyakov – String. Therefore, Conformal Field Theory Methods never play even the slightest role because thereis never any need to introduce and eventually fix any worldsheet metric. Notice also that by definition we only consider representations without Virasoro and Lorentz anomalies, the central charge is zero by definition. Finally, in the modern framework of QFT [4] there is no need for mathematically ill – defined objects such as negative norm states (ghosts) in the Gupta – Bleuler quantization procedure so that all our representations will be ghost – free by definition.

The above mentioned three frameworks can now be combined as follows: The **Pohlmeyer Charges** provide the algebra to be quantized, **AQFT** provides us with techniques to construct representations and finally **LQG** equips us with methods to obtain representations of the algebra of invariants from diffeomorphism invariant representations of kinematical (not gauge invariant) observables.

In this paper we set up the general framework for the representation theory of the closed bosonic string on flat target space and present a non – trivial solution thereof. This solution exists in any target space dimension and has no tachyons. While fermionic worldsheet degrees of freedom are certainly needed for

phenomenological reasons, our solution shows that supersymmetry is not required in all representations of quantum string theory (the degrees of freedom do not necessarily form a supersymmetry multiplett).

One may ask how these celebrated predictions of string theory, 1. a critical dimension of $D = 26$ for the bosonic string and $D = 10$ for the superstring are circumvented. The answer is very simple: In the sense of our defintion, the representation used in ordinary string theory is rather unnatural both from the point of view of AQFT and and LQG: From the point of view of AQFT, the usual Fock representation of string theory does not manifestly define a positive linear functional on the corresponding $^*-$algebra. It is therefore not surprising that the "no – ghost theorem" only holds in critical dimensions and that oursolution to a manifestly ghost – free problem works in any dimension. From the LQG point of view on the other hand the implementation of the constraints in ordinary string theory is rather unnatural because a) one works in a particular worldsheet gauge thus breaking worldsheet diffeomorphism invariance down to the conformal symmtries of the flat worldsheet metric and b) the contraints are implemented only weakly rather than strongly so that one does not perform an honest Dirac quantization. It is therefore not surprising that one usually finds a central charge and that in contrast in our framework CFT methods never play any role as we never have to fix a worldsheet metric. Finally the tachyon in ordinary bosonic string theory is a direct consequence of an ultraviolet divergence in one of the Virasoro generators which is explicitly avoided in LQG. It is therefore not too surprising that we do not find a tachyon.

One can read the results of the present paper partly positively and partly negatively. On the negative side one should notice that, from the purely mathematical point of view, our solution demonstrates that neither a critical dimension nor supersymmetry is a prediction of string theory. Rather, these notions are features of the particular Fock representation of current string theory which is just one solution among possibly zillions of others of the representation problem. Of course, one must show that there are solutions which are physically acceptable from the pheomenological point of view. On the positive side one should notice that the existence of our solution is very encouraging in view of of the fact that solutions with tachyons are unstable and supersymmetric ones without tachyons have yet to be shown to be consistent with phenomenology. There are possibly an infinite number of other stable solutions, including fermionic degrees of freedom, some of which might be closer to the usual Fock representation of string theory than the one we will give in this paper but also much simpler, especially in lower dimensions, which has obious advantages for model building. Indeed, we encourge string theorists and algebraic quantum field theorists to look at the string more abstractly from the algebraic point of view and to systematically develop its representation theory.

The present paper is organized as follows:
Sections two through five merely summarize background material on the classical Nambu – Goto string, its theory of invariants, AQFT and LQG respectively. We have included this material for the benefit of readers from various backgrounds in order to make the paper accessible to a wide audience. These sections can be safely skipped by the experts. The main results of this paper are contained in section six which we summarize once more in section seven. More in detail, here is what we will do:

In section two we recall the canonical formulation of the Nambu – Goto string, the actual geometrical object under consideration. The Polyakov string usually employed in string theory introduces an auxiliary worldsheet metric which is locally pure gauge due to the worldsheet diffeomorphism invariance and an additional Weyl invariance which is absent in the Nambu – Goto formulation. We thus emphasize that there is never any worldsheet metric to be discussed and that Weyl invariance never appears in our formulation. The only local symmetry group is the diffeomorphism (or reparameterization) group Diff(M) of the two – dimensional worldsheet M. We never gauge fix that symmetry in contrast to usual string theory in the Polyakov formulation where one usually uses the conformal gauge which allows to fix the worldsheet metric g to be locally flat η. This gauge fixes the Weyl invariance of the Polyakov string, however, it only partially fixes worldsheet diffeomorphism invariance since conformal symmetries $\varphi \in \mathrm{Conf}_\eta(M) \subset Diff(M)$

(with respect to the flat worldsheet metric η) are still allowed. This residual symmetry is the reason for the importance of conformal QFT techniques in usual string theory, however, in our manifestly Diff(M) – invariant formulation such techniques never play any role.

After having analyzed the Nambu – Goto string as a constrained dynamical system á la Dirac, in section three we recall the theory of the **Pohlmeyer Charges**. The string on flat target spaces turns out to be a completely integrable two – dimensional system and the **Pohlmeyer Charges** are nothing else than the invariants constructed from the corresponding monodromies via Lax pair methods. It is these charges that we want to study interesting representations of.

In section four we recall elements from Algebraic Quantum Field Theory (AQFT). In particular, we describe how cyclic representations of a given $^*-$algebra $\mathfrak{A}$ arise via the Gel'fand – Naimark – Segal (GNS) construction once a positive linear functional (a state) ω is given. This is the same construction that underlies the Wightman reconstruction theorem [11] (reconstructing a Hilbert space from a set of $n-$point functions subject to the positivity requirement). Moreover, if a symmetry group acts on $\mathfrak{A}$ as a group of automorphisms and if ω is invariant then the symmetry group can be implemented as a group of unitary operators on the GNS Hilbert space *without anomalies*.

In section five we recall basics from Loop Quantum Gravity (LQG). In particular, we review the background independent canonical quantization programme which aims at constructing the physical Hilbert space, on which the quantum constraints are identically satisfied, from a given kinematical representation of an algebra of non – gauge invariant operators. The reason for not considering the algebra of invariants right away is that the complete set of invariants is rarely known explicitly for a sufficiently complicated theory (e.g. General Relativity). However, if the kinematical algebra separates the points of the full phase space then the invariant algebra is contained (possibly as a limit) in the kinematical algebra and hence the kinematical representation is a representation of the invariants as well, generically with quantum corrections. The kinematical representation is admissable, however, only if the physical representation induced from it still carries a representation of the invariants. It is in this step that non – trivial regularization techniques come into play.

In section six we collect the results of sections three, four and five to formulate the general representation problem of the closed bosonic string on flat target space. We then present a non – trivial solution to it. Basically, we found a worldsheet diffeomorphism invariant and target space Poincaré invariant state for a kinematical Weyl algebra $\mathfrak{A}$ which contains the **Pohlmeyer Charges** as a limit. We see here how LQG, AQFT and the theory of integrable systems click together: LQG provides a suitable Weyl algebra on which the worldsheet symmetries act as automorphisms, AQFT provides tools to construct representations thereof and finally the theory of integrable systems provides us with the invariant **Pohlmeyer Charges** which can be defined in our representation as well – defined operators and whose vacuum expectation values play the role of *gauge invariant Wightman functions*. In this representation Weyl invariance never arises, worldsheet diffeomorphism invariance and target space Poincaré invariance are exact symmetries without anomalies (central charges), ghosts (negative norm states) never arise and all physical states turn out to be of non – negative mass so that there is no tachyon.

In section seven we conclude, compare our results with ordinary string theory and repeat their consequences.

It should be emphasized that in this paper we heavily exploited that the target space is flat and hence we only need a minimal amount of the techniques of LQG. The full power of LQG techniques however comes into play when we discuss curved target spaces and higher p – brane theories such as the (super –)membrane [12] which, in contrast to string theory, is an interacting theory even on flat target spaces. These issues will be discussed in our companion paper [13] which overlaps in part with the pioneering work [14] but, as the present paper, departs from it in most aspects. See also [15] for a different new approach to string theory using a modification of the Lorentz group.

2 The Nambu – Goto String

In introductory texts to string theory [1] the Nambu – Goto string is barely mentioned, one almost immediately switches to the Polyakov string whose corresponding action has the same set of classical solutions as the Nambu – Goto action. The advantage of the Polyakov action is that it is bilinear in the string degrees of freedom on flat target space, however, this comes at the price of introducing an additional Weyl invariance and an auxiliary worldsheet metric. From the point of a geometer this classical reformulation of the geometerical object, the string, as a theory of scalars interacting with topological gravity is rather unnatural. We thus review below the canonical formulation of the Nambu – Goto action, especially for the benefit of readers without much knowledge of string theory. Experts can safely skip this section. See [16, 17] for more details.

Let us in fact consider the bosonic $p-$brane with $p \geq 2$. Its action is defined in terms of an embedding $X:\ M \to T;\ y \mapsto X^\mu(y)$ from a $p-$dimensional worldsheet M to a $D-$dimensional target space T and a target space metric tensor field $X \mapsto \eta_{\mu\nu}(X)$ of Minkowskian signature $(-1,1,..,1)$ (the case of Euclidean signature is treated in the companion paper [13]). Here we take the worldsheet coordinates $y^\alpha,\ \alpha=0,..,p-1$ to be dimension – free while the target space coordinates $X^\mu, \mu=0,..,D-1$ have dimension of length. The bosonic $p-$brane action is nothing else than the volume of M as measured by η, that is,

$$S[X] := -\frac{1}{\alpha'} \int_M d^p y\ \sqrt{-\det([X^\ast\eta](y))} \tag{2.1}$$

Here α' is called the $p-$brane tension and its has dimensions such that $\hbar\alpha' =: \ell_s^p$ has dimensions of cm^p. In order that X be an embedding, the vector fields $\partial/\partial y^\alpha$ must be everywhere linearly independent on M. Furthermore, the pull-back metric on M given by $X^\ast\eta$ is supposed to have everywhere Minkowskian signature.

We proceed to the canonical analysis. We assume that $M=\mathbb{R}\times\sigma$ where σ is a $(p-1)-$dimensional manifold of fixed but arbitrary topology. The momentum canoniclly conjugate to X^μ is given by

$$\pi_\mu(y) := \alpha' \frac{\delta S}{\delta \dot{X}^\mu(y)} = -\sqrt{-\det(X^\ast\eta)(y)}([(X^\ast\eta)(y)]^{-1})^{t\alpha}\eta_{\mu\nu}(X(y))X^\nu_{,\alpha}(y) \tag{2.2}$$

Introducing temporal and spatial coordinates $y=(t,x)$ we have the elementary Poisson brackets

$$\{X^\mu(t,x),X^\nu(t,x')\} = \{\pi_\mu(t,x),\pi_\nu(t,x')\} = 0,\ \ \{\pi_\mu(t,x),X^\nu(t,x')\} = \alpha'\delta_\mu^\nu\delta(x,x') \tag{2.3}$$

In the sense of Dirac's analysis of constrained systems the Legendre transform (2.2) is not onto, the Langrangean in (2.1) is singular and we arrive at the p primary constraints

$$\begin{aligned} D_a &:= \pi_\mu X^\mu_{,a} = 0 \\ C &:= \frac{1}{2}[\eta^{\mu\nu}\pi_\mu\pi_\nu + \det(q)] = 0 \end{aligned} \tag{2.4}$$

where

$$q_{ab}(y) = [X^\ast\eta]_{ab}(y) \tag{2.5}$$

and $a,b,..=1,..,p-1$. Notice that η maybe an arbitrary curved metric. The constraint D_a is often called spatial diffeomorphism constraint in the LQG literature because its Hamiltonian vector field generates spatial diffeomorphisms of σ. Likewise, the constraint C is called the Hamiltonian constraint because, on the solutions to the equations of motion, it generates canonically temporal reparameterizations. See our companion paper [13] for a more detailed elaboration on this point.

These p constraints are in fact first class so that there are no secondary constraints. Indeed we find the **Hypersurface Deformation Algebra** $\mathfrak{H}$

$$\begin{aligned}
\{D(\vec{N}), D(\vec{N}')\} &= \alpha' D(\mathcal{L}_{\vec{N}} \vec{N}') \\
\{D(\vec{N}), C(N')\} &= \alpha' C(\mathcal{L}_{\vec{N}} N') \\
\{C(N), C(N')\} &= \alpha' \int_\sigma d^{p-1}x \; (N_{,a}N' - NN_{,a}) \, [\det(q) q^{ab}] D_b
\end{aligned} \tag{2.6}$$

where $\mathcal{L}$ denotes the Lie derivative and where we have smeared the constraints with smooth test functions of rapid decrease, that is, $C(N) = \int_\sigma d^{p-1}xNC$ and $D(\vec{N}) = \int_M d^{p-1}xN^aD_a$.

Notice that precisely when $p = 2$ we have that $\det(q)q^{ab} = 1$ is a constant. Thus, among all $p-$brane actions the string is singled out by the fact that its constraints form an honest Lie Poisson algebra. For $p > 2$ we get nontrivial structure functions just like for General Relativity. We will strongly exploit this fact for the rest of the paper for which we restrict our attention to the case $p = 2$ and η is the target spaceMinkowski metric $\eta = \mathrm{diag}(-1, 1, .., 1)$. See our companion paper [13] for the more general case.

Notice that throughout our analysis we have not introduced a worldsheet metric and there is no Weyl symmetry at all. The only local worldsheet symmetry is Diff(M). Also we work manifestly gauge free, that is, we never (partially) fix Diff(M), there is no conformal gauge or anything like that. Finally we want to construct a true Dirac quantization of the system, hence we do not introduce any gauge such as the lightcone gauge in order to solve the constraints classically. This will have the advantage to preserve manifest Lorentz invariance in all steps of our construction.

3 Pohlmeyer Charges

In the rest of the paper we focus on the closed bosonic string, i.e. $p = 2$ and $\sigma = S^1$ on flat Minkowski space. We take our spatial coordinate x to be in the range $[0, 2\pi)$. All functions in what follows are periodic with period 2π unless specified otherwise.

From the classical point of view we must find the (strong) Dirac observables of the system, that is, functions on the phase space coordinatized by X, π which have (strongly) vanishing Poisson brackets with the constraints. While the string is a parameterized system so that the Hamiltonian defined by the Legendre transform vanishes identically, the initial value constraints do generate arbitrary time reparameterizations and in this sense the problem of finding the Dirac observables is closely related to finding the integrals of motion of a dynamical system [18]. This leads us to the theory of integrable systems [19] and the very powerful techniques that have been developed for those, especially in two dimensions. Pohlmeyer et. al. have shown in an impressive series of papers [7, 8, 9, 10] that the string is completely integrable. Since to the best of our knowledge these important works are basically unknown even among string theorists, in what follows we will summarize the basics of these developments, following the beautiful thesis [20].

3.1 Automorphisms of Gauge – and Symmetry Transformations

The first step is to introduce an equivalent set of constraints called the two Virasoro constraints

$$V_\pm(u) := \pm\frac{1}{2\alpha'}\int_{S^1} dx \; u(C \pm D) \equiv \pm\frac{1}{2\alpha'}\int_{S^1} dx \; \xi\eta_{\mu\nu}Y^\mu_\pm Y^\nu_\pm \tag{3.1}$$

where u is a smearing function and

$$Y^\mu_\pm := \eta^{\mu\nu}\pi_\nu \pm X^{\mu\prime} \tag{3.2}$$

Here and in what follows a prime denotes derivation with respect to x while a dot denotes derivation with respect to t. The advantage of (3.1) over (2.4) is that the constraint algebra simplifies to

$$\{V_\pm(u), V_\mp(v)\} = 0, \;\; \{V_\pm(u), V_\pm(v)\} = V_\pm(s(u, v)) \tag{3.3}$$

where $s(u,v)=u'v-uv'$. Thus the constraint algebra can be displayed as the direct sum of two algebras each of which is isomorphic to the Lie algebra of the diffeomorphism group $\mathrm{Diff}(S^1)$ of the circle. These two algebras are, however, not the same as yet a third copy of $\mathrm{diff}(S^1)$ generated by $D=V_+ + V_-$ which generates diffeomorphisms of the circle for all phase space functions while $V_\pm$ do that only for functions of $Y_\pm$. Hence, there are three different diffeomorphism groups of the circle at play which have to be cleanly distinguished.

The important functions $Y_\pm(f) := \int_{S^1} dx\, f_\mu\, Y^\mu_\pm$ themselves obey the following algebra

$$\{Y_\pm(f), Y_\mp(g)\}=0,\ \ \{Y_\pm(f),Y_\pm(g)\}=\pm\alpha'\eta^{\mu\nu}\int_{S^1} dx\ (f'_\mu g_\nu - f_\mu g'_\nu) \tag{3.4}$$

From the geometrical point of view the $Y_\pm$ are one forms on S^1 and the f are scalars while the u,v are vector fields. In one dimension, $p-$times covariant and $q-$times contravariant tensors are the same thing as scalar densities of weight $p-q$, hence all integrals are over scalar densities of weight one which are spatially diffeomorphism invariant. However, the Hamiltonian vector fields corresponding to the $V_\pm(\xi)$ only act on the $Y_\pm$ not on the u,v,f,f'. Specifically we have

$$\{V_\pm(u),Y_\mp(f)\}=0,\ \ \{V_\pm(u),Y_\pm(f)\}=Y_\pm(uf') \tag{3.5}$$

Hence the Virasoro generators $V_\pm(u)$ act on $Y_\pm$ by infinitesimal diffeomorphisms while they leave $Y_\mp$ invariant. Consider the Hamiltonian flow of $V_\pm(u)$ defined by

$$[\alpha^\pm_u(t)](F) := \sum_{n=0}^\infty \frac{t^n}{n!}\{V_\pm(u),F\}_{(n)} \tag{3.6}$$

where F is any smooth function on phase space and the repeated Poisson bracket is inductively defined by $\{G,F\}_{(0)}=F,\ \{G,F\}_{(n+1)}=\{G,\{G,F\}_{(n)}\}$. It is easy to check that

$$[\alpha^\pm_u(t)](Y_\mp(x))=Y_\mp(x),\ \ [\alpha^\pm_u(t)](Y_\pm(x))=([\varphi^u_t]^* Y_\pm)(x) \tag{3.7}$$

where $t\mapsto\varphi^u_t$ is the one parameter group of diffeomorphisms of S^1 defined by the integral curves $t\mapsto c^u_x(t)$ through x of the vector field u on S^1, that is, $\varphi^u_t(x)=c^u_x(t)$. Here φ^* denotes the pull – back of $p-$forms.

Since the Hamiltonian flow of a smooth phase space function defines an automorphism of the Poisson algebra of smooth functions on phase space, we obtain for any smooth function of the functional form $F=F[Y_+,Y_-]$ that

$$[\alpha^+_u(t)](F)=F[(\varphi^u_t)^*Y_+,Y_-],\ \ [\alpha^-_u(t)](F)=F[Y_+,(\varphi^u_t)^*Y_-] \tag{3.8}$$

Writing $\alpha^\pm_{\varphi^u_t} := \alpha^\pm_u(t)$ we may extend (3.8) to all elements φ of $\mathrm{Diff}(S^1)$

$$[\alpha^+_\varphi](F)=F[\varphi^*Y_+,Y_-],\ \ [\alpha^-_\varphi](F)=F[Y_+,\varphi^*Y_-] \tag{3.9}$$

Besides this local gauge freedom of the string we also have global Poincaré symmetry. The generators of infinitesimal translations and Lorentz transformations respectively are given by

$$\begin{aligned} p_\mu &= \frac{1}{\alpha'}\int_{S^1} dx\ \pi_\mu \\ J^{\mu\nu} &= \frac{1}{\alpha'}\int_{S^1} dx\ [X^\nu\eta^{\mu\rho}\pi_\rho - X^\mu\eta^{\nu\rho}\pi_\rho] \end{aligned} \tag{3.10}$$

It is straightforward to check that $p_\mu, J^{\mu\nu}$ have vanishing Poisson brackets with the $V_\pm(u)$ and thus are strong Dirac observables. Moreover, we have

$$\{p_\mu, Y^\nu_\pm(x)\}=0,\ \ \{J^{\mu\nu},Y^\rho_\pm(x)\}=(\eta^{\mu\rho}Y^\nu_\pm - \eta^{\nu\rho}Y^\mu_\pm)(x) \tag{3.11}$$

Hence for the corresponding Hamiltonian flows we obtain

$$\begin{aligned}\alpha_a(Y_\pm) &:= \sum_{n=0}^{\infty} \frac{1}{n!} \{a^\mu p_\mu, Y_\pm\}_{(n)} = Y_\pm \\ \alpha_\Lambda(Y_\pm) &:= \sum_{n=0}^{\infty} \frac{1}{n!} \{\Lambda_{\mu\nu} J^{\mu\nu}, Y_\pm\}_{(n)} = \exp(\Lambda_{\mu\nu}\tau^{\mu\nu}) \cdot Y_\pm \end{aligned} \tag{3.12}$$

where Λ is an antisymmetric matrix and $\tau^{\mu\nu}$ are appropriate basis elements of $so(1, D-1)$. Denoting $L = \exp(\Lambda_{\mu\nu}\tau^{\mu\nu})$ the automorphisms extend to smooth functions of $Y_\pm$ as

$$\alpha_a(F) = F, \quad \alpha_L(F) = F[L \cdot Y_+, L \cdot Y_-] \tag{3.13}$$

3.2 Algebra of Invariants

This finishes our analysis of the actions of the constraints and symmetry generators on the $Y_\pm$. We will now construct Dirac observables $F[Y_+, Y_-]$ by using the theory of integrable systems. The idea is to use the method of Lax pairs, that is, one reformulates the equations of motion for $Y_\pm$ as a matrix equation of the form $M_{,\alpha} = [A_\alpha, M]$ where $M(t,x)$ is an $N \times N$ matrix and A_α is a "connection", that is, a one form on $\mathbb{R} \times S^1$ with values in $GL(N, \mathbb{C})$. The integrability condition for the Lax pair L, A is the zero curvature equation $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta] = 0$ and one looks for L, A in such a way that $F_{\alpha\beta} = 0$ is equivalent to the equations of motion. Given such a setup, it follows that the functions $\mathrm{Tr}(L^n)$, $n = 1, .., N$ are constants of the motion ($n > N$ leads to algebraically dependent invariants due to the theorem of Hamilton – Caley).

We proceed to the details. The "Hamiltonian" of the string is given by

$$H(u,v) = \frac{1}{\alpha'}(C(u) + D(v)) = V_+(v+u) + V_-(v-u) \tag{3.14}$$

where u, v are arbitrary test functions. "Time" evolution is defined by

$$\dot{Y}_\pm := \{H(u,v), Y_\pm\} = [(v \pm u) Y_\pm]' \tag{3.15}$$

Notice that the "time" evolution depends on u, v and it will be our task to show that our final invariants constructed by the Lax pair method do not depend on u, v.

Let τ_I be a basis of $GL(n, \mathbb{R})$ and T^I_μ some complex valued matrices. We define the following connection

$$A^\pm_x := Y^\mu_\pm T^I_\mu \tau_I =: Y^\mu_\pm T_\mu, \quad A^\pm_t := (v \pm u) A^\pm_x \tag{3.16}$$

The zero curvature condition reproduces the equations of motion

$$F^\pm_{tx} = \partial_t A^\pm_x - \partial_x A^\pm_t + [A^\pm_t, A^\pm_x] = \partial_t A^\pm_x - \partial_x A^\pm_t = \partial_t A^\pm_x - \{H(u,v), A^\pm_x\} = 0 \tag{3.17}$$

Given a curve c on $M = \mathbb{R} \times S^1$ we define its holonomy as the usual path ordered product

$$h_c(A^\pm) = \mathcal{P} \exp(\int_c dx\ A^\pm) \tag{3.18}$$

where $\mathcal{P}$ denotes path ordering with the lowest parameter value to the outmost left. Given a parameterization of the path $[0, b] \to c;\ s \mapsto c(s)$ the holonomy is the unique solution to the ordinary differential equation

$$\frac{d}{ds} h_{c_s}(A^\pm) = h_{c_s}(A^\pm) A^\pm_\alpha(c(s)) \dot{c}^\alpha(s), \quad h_{c_0}(A^\pm) := 1_N, \quad h_{c_b}(A^\pm) \equiv h_c(A^\pm) \tag{3.19}$$

where $c_s = c([0, s])$.

Supposing that the curvature of A vanishes, that is, that A is flat, it is clear that for loops c on $\mathbb{R} \times S^1$ only those which are not contractible lead to a non − trivial value of $h_c(A)$. Among the non contractible loops of winding number one all are homotopic to the loop $c_{t,x} = \{t\} \times [x, x + 2\pi)$ in the worldsheet time slice t =const. winding once around the cylinder with starting point x. In the theory of integrable systems, the holonomy of that loop is called the monodromy matrix

$$h_{t,x}(A^{\pm}) := h_{c_{t,x}}(A^{\pm}) \tag{3.20}$$

Notice that (3.20) is not a periodic function of x even if the connection A is flat.

Consider an arbitrary interior point x_0 of $c_{t,x}$. This point subdivides the loop into two edges $e_{t,x} = \{t\} \times [x_0, x]$ and $e'_{t,x} = \{t\} \times [x_0, x + 2\pi]$, that is, $c_{t,x} = e_{t,x}^{-1} \circ e'_{t,x}$. Hence we have by basic properties of the holonomy

$$h_{t,x}(A^{\pm}) := (h_{e_{t,x}}(A^{\pm}))^{-1} h_{e'_{t,x}}(A^{\pm}) \tag{3.21}$$

By the very definition of the holonomy of a connection along a path (3.19) we find with the parameterization $c_{t,x}(s) = x + s,\ s \in [0, 2\pi)$

$$\partial_x h_{t,x}(A^{\pm}) = [h_{t,x}(A^{\pm}), A_x(t,x)] \tag{3.22}$$

On the other hand we have

$$\begin{aligned}
\partial_t h_{t,x}(A^{\pm}) &= \lim_{\epsilon \to 0} \frac{1}{\epsilon}(h_{t+\epsilon,x}(A^{\pm}) - h_{t,x}(A^{\pm})) \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon}(h_{e_{\epsilon,t,x}}(A^{\pm})^{-1}[h_{e_{\epsilon,t,x}}(A^{\pm}) h_{t+\epsilon,x}(A^{\pm})] - h_{t,x}(A^{\pm})) \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon}(h_{e_{\epsilon,t,x}}(A^{\pm})^{-1}[h_{t,x}(A^{\pm}) h_{e_{\epsilon,t,x}}(A^{\pm})] - h_{t,x}(A^{\pm})) \\
&= [h_{t,x}, A_t^{\pm}(t,x)]
\end{aligned} \tag{3.23}$$

where $e_{\epsilon,t,x}(s) = (t+s, x),\ s \in [0, \epsilon]$. Here we have used that the loop $c_{t,x} \circ e_{\epsilon,t,x} \circ c_{t+\epsilon,x}^{-1} \circ e_{\epsilon,t,x}^{-1}$ is contractible and the zero curvature condition in the third step.

Conversely, postulating equations (3.22) and (3.23) we discover, using the Jacobi identity

$$2\partial_{[\alpha}\partial_{\beta]} h_{t,x}(A^{\pm}) = [h_{t,x}(A^{\pm}), F^{\pm}_{\alpha,\beta}] \tag{3.24}$$

where $F^{\pm}$ is the curvature of $A^{\pm}$. Hence the zero curvature condition is the integrability condition for the equations $\partial_\alpha h_{t,x}(A^{\pm}) = [h_{t,x}(A^{\pm}), A^{\pm}_\alpha(t,x)]$.

We now claim that for any $n \leq N$ the functions

$$T^n_{t,x}(A^{\pm}) := \mathrm{Tr}_N([h_{t,x}(A^{\pm})]^n) \tag{3.25}$$

are independent of both x, t. This follows immediately from (3.22) and (3.23) as well as the cyclicity of the trace. What is, however, even more remarkable is that (3.25) has vanishing Poisson brackets with $H(u,v)$ *for all* u, v even though $A^{\pm}_t = (v \pm u) A^{\pm}_x$ depends explicitly on u, v. The reason for this is that $A^{\pm}_x$ is actually independent of u, v and that the dependence of the time evolution of $h_{t,x}(A^{\pm})$ consists just in a prefactor

$$\partial_t h_{t,x}(A^{\pm}) = (v \pm u)\partial_x h_{t,x}(A^{\pm}) \tag{3.26}$$

and the second factor in (3.26) is independent of u, v. More precisely we have

$$\begin{aligned}
\{H(u,v), h_{t,x}(A^{\pm})\} &= \int_{S^1} dy\ \{H(u,v), Y^\mu_\pm(t,y)\} \frac{\delta h_{t,x}(A^{\pm})}{\delta Y^\mu_\pm(t,y)} \\
&=: \int_{S^1} dy\ [\partial_t Y^\mu_\pm(t,y)] \frac{\delta h_{t,x}(A^{\pm})}{\delta Y^\mu_\pm(t,y)} \\
&= \partial_t h_{t,x}(A^{\pm})
\end{aligned} \tag{3.27}$$

One can of course verify by direct computation from the expression (3.18) that the $T^n(A^\pm)$ are Dirac observables for the string.

Since we can trade the power $n \le N$ for considering increasing rank N of the matrices τ_I the only interesting invariant is the generating functional

$$Z^T_\pm := T^1(A^\pm) = N + \sum_{n=1}^{\infty} Z_\pm^{\mu_1..\mu_N} \; \frac{\mathrm{Tr}_N(T_{\mu_1}..T_{\mu_N})}{N} \tag{3.28}$$

where $T^\mu = T^\mu_I \tau^I$ and we have used the expansion (3.18). It is understood that N can be arbitrarily large and that the complex matrices $T^\mu \in GL(N,\mathbb{C})$ are freely specificable while $Z^T_\pm$ is an invariant. Thus we conclude that the expansion coefficients are themselves invariants. These are the **Pohlmeyer Charges**

$$\begin{aligned} Z_\pm^{\mu_1..\mu_N} &= [R^{\mu_1..\mu_N}(x) + R^{\mu_2..\mu_N\mu_1}(x) + .. + R^{\mu_N\mu_1..\mu_{N-1}}(x)] \\ R_\pm^{\mu_1..\mu_N}(x) &= \int_x^{x+2\pi} dx_1 \int_{x_1}^{x+2\pi} dx_2 \, .. \int_{x_{N-1}}^{2+2\pi} dx_N \; Y_\pm^{\mu_1}(x_1) \, .. \, Y_\pm^{\mu_N}(x_N) \\ &=: \int_{x\le x_1 \le .. \le x_N \le x+2\pi} d^N x \; Y_\pm^{\mu_1}(x_1) \, .. \, Y_\pm^{\mu_N}(x_N) \end{aligned} \tag{3.29}$$

The functionals $R^{(N)}_\pm(x)$ depend explicitly on x, they are not invariants. It is only after cyclic summation $C_N \cdot R^{(N)}_\pm = Z^{(N)}_\pm$ that they become invariants.

The **Pohlmeyer Charges** exepectedly form a complicated, closed subalgebra of the full Poisson algebra of the string. We just quote the result and refer the reader to the literature [7, 8, 9, 10, 20]. One finds after a lot of algebra

$$\{Z_\pm^{\mu_1..\mu_M}, Z_\mp^{\nu_1..\nu_N}\} = 0 \tag{3.30}$$

$$\begin{aligned} &\{Z_\pm^{\mu_1..\mu_M}, Z_\pm^{\nu_1..\nu_N}\} = \mp 2\alpha' \sum_{m=1}^{M} \sum_{n=1}^{N} \eta^{\mu_m\nu_n} \times \\ &\times [S_{\mu_{m+2}..\mu_{m-1};\nu_{n+1}..\nu_{n-2}} \cdot Z_\pm^{\mu_{m+1}..\mu_{m-1}\nu_{n+1}..\nu_{n-1}} - S_{\mu_{m+1}..\mu_{m-2};\nu_{n+2}..\nu_{n-1}} \cdot Z_\pm^{\nu_{n+1}..\nu_{n-1}\mu_{m+1}..\mu_{m-1}}] \end{aligned}$$

Here the symbol $S_{a_1..a_m;b_1..b_n}$ denotes the shuffle operator on multi – indexes, that is, it imposes summation over all permutations of the $a_1,..,a_m,b_1,..,b_n$ such that a_k is always to the left of a_l for $1 \le k \le l \le m$ and such that b_k is always to the left of b_l for $1 \le k \le l \le n$, ie. the sequence of the indices $a_1,..,a_m$ and $b_1,..,b_n$ remains unaltered. They even form a $^*-$algebra, namely

$$\begin{aligned} (Z_\pm^{\mu_1..\mu_M})^* &:= \overline{Z_\pm^{\mu_1..\mu_M}} = Z_\pm^{\mu_1..\mu_M} \\ Z_\pm^{\mu_1..\mu_M} \;\; Z_\pm^{\mu_{M+1}..\mu_{M+N}} &= C_N \cdot [S_{\mu_1..\mu_M;\mu_{M+1}..\mu_{M+N-1}} \cdot Z_\pm^{\mu_1..\mu_{M+N}}] \end{aligned} \tag{3.31}$$

where the cyclic summation acts on the $\mu_1,..,\mu_N$ only. Notice the relations $p^\mu = Z^\mu_\pm$ and

$$\{p^\mu, Z_\pm^{\mu_1..\mu_N}\} = \{Z^\mu_\mp, Z_\pm^{\mu_1..\mu_N}\} = 0 \tag{3.32}$$

As one can show, together with $J^{\mu\nu}$ the invariants $Z_\pm$ provide a **complete** system of invariants for the string in the sense that one can reconstruct $X^\mu(t,x)$ up to gauge transformations (parameterizations) and up to translations in the direction of p^μ. Unfortunately, the invariants $Z_\pm$ are not algebraically independent, that is, they are overcomplete because there are polynomial relations between them. However, it is possible to construct so – called *standard invariants* [7, 8] of which all the $Z_\pm$ are polynomials. Although we do not need them for what follows we will briefly sketch their definition in order to summarize the state of the art of Pohlmeyer's algebraic approach to string theory.

The first step is to consider the logarithm of the monodromy matrix by means of the identity

$$h_{t,x}(A^\pm) = \exp(\ln(h_{t,x}(A^\pm))) = \sum_{k=0}^{\infty} \frac{1}{k!} [\ln(h_{t,x}(A^\pm))]^k \tag{3.33}$$

Close to $T_\mu = 0$ or $h_{t,x}(A^\pm) = 1_N$ we can expand the logarithm as for $|x-1| < 1$

$$\ln(x) = \ln(1-(1-x)) = -\sum_{n=1}^{\infty} \frac{1}{n}(1-x)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n \tag{3.34}$$

so that

$$\frac{1}{k!}[\ln(x)]^k = \sum_{n=k}^{\infty} c_{kn}(x-1)^n \tag{3.35}$$

for certain coefficients c_{kn} satisfying $c_{kn} = 0$ for $k > n$. Noting that

$$h_{t,x}(A^\pm) - 1_N = \sum_{M=1}^{\infty} R_\pm^{\mu_1..\mu_M} T_{\mu_1}..T_{\mu_M} \tag{3.36}$$

we see that the coefficients $R_{\pm,k}$ of

$$\frac{1}{k!}[\ln(h_{t,x}(A^\pm))]^k = \sum_{n=k}^{\infty} R_{\pm,k}^{\mu_1..\mu_n} T_{\mu_1}..T_{\mu_n} \tag{3.37}$$

are given by polynomomials of the tensors $R_\pm$ of tensor rank $1 \leq M \leq n$ whose tensor ranks add up to n. Specifically they are given by the multi – shuffle – sums

$$R_{\pm,k}^{\mu_1..\mu_n} = \sum_{m=k}^{n} c_{km} \sum_{0<a_1<..<a_{m-1}<n} R_\pm^{\mu_1..\mu_{a_1}}\; R_\pm^{\mu_{a_1+1}..\mu_{a_2}}\; ..\; R_\pm^{\mu_{a_{m-1}+1}..\mu_n} \tag{3.38}$$

They are therefore called the *homogeneous* tensors. The *truncated* tensors are simply

$$R_{\pm,t}^{\mu_1..\mu_n} := R_{\pm,k=1}^{\mu_1..\mu_n} \tag{3.39}$$

Their importance lies in the fact that one can write the $R_\pm$ in terms of the $R_{\pm,t}$ namely

$$\begin{aligned} R_\pm^{\mu_1..\mu_n} &= \sum_{k=1}^{n} R_{\pm,k}^{\mu_1..\mu_n} \\ R_{\pm,k}^{\mu_1..\mu_n} &= \frac{1}{k!} \sum_{0<a_1<..<a_{k-1}<n} R_{\pm,t}^{\mu_1..\mu_{a_1}}\; R_{\pm,t}^{\mu_{a_1+1}..\mu_{a_2}}\; ..\; R_{\pm,t}^{\mu_{a_{k-1}+1}..\mu_n} \end{aligned} \tag{3.40}$$

The first relation in (3.40) has a direct analog for the invariants themselves

$$Z_\pm^{\mu_1..\mu_n} = \sum_{k=1}^{n} Z_{\pm,k}^{\mu_1..\mu_n} \text{ where } Z_{\pm,k}^{\mu_1..\mu_n} = C_n \cdot R_{\pm,k}^{\mu_1..\mu_n} \tag{3.41}$$

so that all invariants are polynomials in the truncated tensors.

We notice that the the homogeneous invariants $Z_{\pm,k}^{(n)}$ carry two gradings, the tensor rank $n \geq 0$ and the homogeneity degree $k \geq 0$. Under Poisson brackets these two gradings behave as follows

$$\{Z_{\pm,k}^{(n)}, Z_{\pm,k'}^{(n')}\} = Z_{\pm,k+k'-1}^{(n+n'-2)} \tag{3.42}$$

We can define a new grading degree $L := n-k-1,\ n \geq k$, called the *layer*, which behaves additively under Poisson brackets

$$\{Z_\pm(L), Z_\pm(L')\} = Z_\pm(L+L') \tag{3.43}$$

where $Z_\pm(L)$ is any linear combination of homogeneous invariants $Z_{\pm,k}^{(n)}$ such that $n-k-1 = L$. Notice that the vector space V_L of the $Z_\pm(L)$ is infinite dimensional for each finite L.

Consider now the layer V_L. It contains the following *standard invariants*

$$Z^{0,..,0,a,\mu_2,..,\mu_{l+1},b}_{\pm,L+1} \tag{3.44}$$

where $a,b=1,..D-1$ are spacelike tensor components and the first L tensor components are zero (timelike). One can show [20] that these standard invariants, together with p_μ, define an *algebraic basis* for the string, that is, they are polynomially independent and all other invariants are polynomials of those.

Remarkably, almost all standard invariants of the layers $L\geq 2$ can be obtained by multiple Poisson brackets of the layers $L=0,1$ (the layer $L=-1$ are the momentum components p_μ which are the central elements of the invariant algebra). In each layer with L odd one finds so – called *exceptional elements*

$$Z^{0\mu 0..0\nu}_{\pm,2} \tag{3.45}$$

with L zeroes in between μ,ν which cannot be generated via Poisson brackets from the $L=0,1$ standard invariants. In an incredible computational effort it has been verified that up to layer $L=7$ in $D=3$ that the exceptional elements are the only such invariants which cannot be generated. It is assumed, but has not been proved yet, that the *quadratic generation hypothesis* holds, namely that the standard invariants of the first two layers $L=0,1$ together with the infinite number of exceptional elements from the odd layers generate the full algebra of invariants via multiple Poisson brackets (the name "quadratic" is due to the fact that the exceptional elements (3.45) have homogeneity degree $k=2$).

Let us assume that the quadratic generation hypothesis holds and denote by $\mathfrak{G}$ the vector space of generators of the full algebra $\mathfrak{Z}$ of invariants. A given invariant in $\mathfrak{Z}$ can, however, be written in many ways as a linear combination of multiple Poisson brackets between elements from $\mathfrak{G}$, for instance, due to the Jacobi identity for the Poisson bracket, but there are even more relations as one can see by computing the number of standard invariants in each layer L and the number of possible Poisson brackets between generators with range in that layer. In other words, the vector space $\mathfrak{G}$ does not generate $\mathfrak{Z}$ freely. Denote by $S(\mathfrak{G})$ the symmetric (i.e. commuting under the tensor product) envelopping Lie algebra of $\mathfrak{G}$ and by $\mathfrak{J}\subset S(\mathfrak{G})$ the ideal generated by these polynomial relations between multiple Poisson brackets. Then $\mathfrak{Z}=S(\mathfrak{G})/\mathfrak{J}$.

Pohlmeyer's programme to quantize the string algebraically now consists in the construction of the universal (i.e. non – commutating under the tensor product) envelopping algebra $\hat{U}(\mathfrak{G})$ of $\mathfrak{G}$ quotiented by a quantum ideal $\widehat{\mathfrak{J}}$. The quantum ideal roughly arises from the classical one by replacing Poisson brackets by commutators divided by $i\hbar$ with symmetric ordering, whereby higher order (in $\hbar$) quantum corrections are allowed. These corrections have to be chosen *self – consistently* in such a way that the number of quantum relations is the same as the number of classical relations in order that both classical and quantum theory have the same number of degrees of freedom. Due to the non – commutativity of quantum theory it might happen that commutators between quantum relations and quantum generators produce *anomalies*, that is, new relations without classical counterpart. Again, in an enormous compuational effort it has been shown in $D=4$ that there are suitable non – anomalous choices for the quantum relations up to layer five.

It appears hopelessly difficult to complete this programme to all layers. However, recently [8] there has been made important progress in this respect: It is possible to identify $\mathfrak{Z}$ as the kernel of a larger algebra $S(\mathfrak{R})$ without relations under a suitable derivation $\partial_x:\ S(\mathfrak{R})\to\mathbb{R}^D\times S(\mathfrak{R})$ where $\mathfrak{R}$ stands for a set of generators (constructed from the truncated tensors) larger than $\mathfrak{G}$. One then quantizes, like in BRST quantization, a quantum derivation δ on the universal envelopping algebra $\hat{U}(\mathfrak{R})$ of $\mathfrak{R}$ and defines the quantum algebra as the kernel of δ. This takes, self-consistently, care of all quantum relations in all layers in *a single stroke.*

We hope to have given a useful and fair introduction to the **Pohlmeyer String**. The outstanding problems in this programme are 1. the proof of the quadratic generation hypothesis and 2. the proof that the final quantum algebra $\widehat{\mathfrak{Z}}$ of invariants has interesting representations. Notice that the approach is completely intrinsic, it is a *quantization after solving the constraints* because one works on the reduced phase space of the unconstrained phase space and seeks a quantization of the corresponding Dirac observables.

For the LQG string we proceed differently, namely we adopt Dirac's programme of *solving the constraints after quantizing*. This means, in particular, that we have to introduce a kinematical, that is, unphysical algebra $\mathfrak{A}$ of observables which contains the **Pohlmeyer Charges** as certain limits. We define the representation theory of $\mathfrak{A}$, taking care of the quantum constraints by methods of LQG and AQFT and then define physically interesting representations as those which support the **Quantum Pohlmeyer Charges**. This presents an alternative route to quantizing $\mathfrak{Z}$.

Let us clarify this difference between the two programmes: In Pohlmeyer's programme one starts directly from the classical Poisson algebra of invariant charges. One then seeks for representations of the corresponding quantum algebra which may involve $\hbar$ corrections. In this intrinsic approach it is important that the quantum corrections are again polynomials of the charge operators because it is only those that one quantizes. Moreover, one must make sure that there are no anomalies in the corresponding quantum ideal. This is a very difficult programme because the charge algebra is very complicated and it is not a priori clear that the infinitely many consistency conditions among the quantum corrections are satisfied. However, modulo establishing the quadratic generation hypothesis, appropriate quantum corrections can be found and one is left with the problem of finding irreducible rpresentations. This is even more difficult given the expectedly difficult structure of the quantum corrections. In this paper we will take a much less ambitious approach which bears a lot of similarity with the proposal of [8]: Namely, we start with a fundamental algebra $\mathfrak{A}$ which is much simpler but also much larger than the extended algebra considered in [8] and seek representations of that. The quantum algebra $\widehat{\mathfrak{Z}}$ is then a derived object. The advantage is that the quantum corrections for $\mathfrak{Z}$ that we necessarily get are not supposed to be generated by elements of $\mathfrak{Z}$ again, rather they may be generated from elements of $\mathfrak{A}$ which is clearly the case. Moreover, the quantum ideal corresponding to $\mathfrak{A}$ is much simpler, in fact it is rather standard in AQFT. This not only simplifies the algebraic problem tremendously but also the representation theory. The only thing that one has to make sure is that the quantum corrections really vanish in the classical limit.

4 Operator Algebras and Algebraic Quantum Field Theory

We include this section only as background material for the unfamiliar reader, see e.g. [21] for further information. Everything that we summarize here is standard knowledge in mathematical physics and can be safely skipped by the experts.

I. *Operator Algebras*

An algebra $\mathfrak{A}$ is simply a vector space over $\mathbb{C}$ in which there is defined an associative and distributive multiplication. It is unital if there is a unit $\mathbf{1}$ which satisfies $a\mathbf{1} = \mathbf{1}a = a$ for all $a \in \mathfrak{A}$. It is a $^*-$algebra if there is defined an involution satisfying $(ab)^* = b^*a^*,\ (a^*)^* = a$ which reduces to complex conjugation on the scalars $z \in \mathbb{C}$.

A Banach algebra is an algebra with norm $a \mapsto ||a|| \in \mathbb{R}^+$ which satisfies the usual axioms $||a+b|| \leq ||a|| + ||b||,\ ||ab|| \leq ||a||\ ||b||,\ ||za|| = |z|\ ||a||,\ ||a|| = 0 \quad \Leftrightarrow \quad a = 0$ and with respect to which it is complete.

A C^*-algebra is a Banach $^*-$algebra whose norm satisfies the C^*-property $||a^*a|| = ||a||^2$ for all $a \in \mathfrak{A}$. Physicists are most familiar with the C^*-algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$.

II. *Representations*

A representation of a $^*-$algebra $\mathfrak{A}$ is a pair $(\mathcal{H}, \pi)$ consisting of a Hilbert space $\mathcal{H}$ and a morphism $\pi:\ \mathfrak{A} \to \mathcal{L}(\mathcal{H})$ into the algebra of linear (not necessarily bounded) operators on $\mathcal{H}$ with common and invariant dense domain. This means that $\pi(za + z'a') = z\pi(a) + z'\pi(a'),\ \pi(ab) = \pi(a)\pi(b),\ \pi(a^*) = [\pi(a)]^\dagger$ where † denotes the adjoint in $\mathcal{H}$.

The representation is said to be faithful if $\text{Ker}\pi = \{0\}$ and non – degenerate if $\pi(a)\psi = 0$ for all $a \in \mathfrak{A}$ implies $\psi = 0$.

A representation is said to be cyclic if there exists a normed vector $\Omega \in \mathcal{H}$ in the common domain of all the $a \in \mathfrak{A}$ such that $\pi(\mathfrak{A})\Omega$ is dense in $\mathcal{H}$. Notice that the existence of a cyclic vector implies that the states $\pi(b)\Omega,\ b \in \mathfrak{A}$ lie in the common dense and invariant domain for all $\pi(a),\ a \in \mathfrak{A}$. A representation is said to be irreducible if every vector in a common dense and invariant (for $\mathfrak{A}$) domain is cyclic.

III. *States*
A state on a $^*-$algebra is a linear functional $\omega :\ \mathfrak{A} \to \mathbb{C}$ which is positive, that is, $\omega(a^*a) \geq 0$ for all $a \in \mathfrak{A}$. If $\mathfrak{A}$ is unital we require that $\omega(\mathbf{1}) = 1$. The states that physicists are most familiar with are vector states, that is, if we are given a representation $(\mathcal{H}, \pi)$ and an element ψ in the common domain of all the $a \in \mathfrak{A}$ then $a \mapsto < \psi, \pi(a)\psi >_{\mathcal{H}}$ evidently defines a state. These are examples of pure states, i.e. those which cannot be written as convex linear combinations of other states. However, the concept of states is much more general and includes what physicists would call mixed (or temperature) states.

IV. *Automorphisms*
An automorphism of a $^*-$algebra is an isomorphism of $\mathfrak{A}$ which is compatible with the algebraic structure. If G is a group then G is said to be represented on $\mathfrak{A}$ by a group of automorphisms $\alpha :\ G \to \text{Aut}(\mathfrak{A});\ g \mapsto \alpha_g$ provided that $\alpha_g \circ \alpha_{g'} = \alpha_{gg'}$ for all $g, g' \in G$. A state ω on $\mathfrak{A}$ is said to be invariant for an automorphism α provided that $\omega \circ \alpha = \omega$. It is said to be invariant for G if it is invariant for all $\alpha_g,\ g \in G$.

The following two structural theorems combine the notions introduced above and are of fundamental importance for the construction and analysis of representations.

Theorem 4.1 (GNS Construction).
Let ω be a state on a unital $^*-$algebra $\mathfrak{A}$. Then there are GNS data $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ consisting of a Hilbert space $\mathcal{H}_\omega$, a cyclic representation π_ω of $\mathfrak{A}$ on $\mathcal{H}_\omega$ and a normed, cyclic vector $\Omega_\omega \in \mathcal{H}_\omega$ such that

$$\omega(a) = < \Omega_\omega, \pi_\omega(a)\Omega_\omega >_{\mathcal{H}_\omega} \tag{4.1}$$

Moreover, the GNS data are determined by (4.1) uniquely up to unitary equivalence.

The name GNS stands for Gel'fand – Naimark – Segal. The idea is very simple. The algebra $\mathfrak{A}$ is in particular a vector space and we can equip it with a sesqui – linear form $< a, b >:= \omega(a^*b)$. This form is not necessarily positive definite. However, by exploiting the Cauchy – Schwarz inequality $|\omega(a^*b)|^2 \leq \omega(a^*a)\omega(b^*b)$ one convinces oneself that the set $\mathfrak{I}_\omega$ consisting of the elements of $\mathfrak{A}$ satisfying $\omega(a^*a) = 0$ defines a left ideal. We can thus pass to the equivalence classes $[a] = \{a + b;\ b \in \mathfrak{I}_\omega\}$ and define a positive definite scalar product by $< [a], [b] >:= \omega(a^*b)$ for which one checks independence of the representative. Since $\mathfrak{I}_\omega$ is a left ideal one checks that $[a]+[b] := [a+b],\ z[a] := [za],\ [a][b] := [ab]$ are well defined operations. Then $\mathcal{H}_\omega$ is simply the Cauchy completion of the vectors $[a]$, the representation is simply $\pi_\omega(a)[b] := [ab]$ and the cyclic vector is just given by $\Omega_\omega := [\mathbf{1}]$. Finally, if $(\mathcal{H}'_\omega, \pi'_\omega, \Omega'_\omega)$ are other GNS data then the operator $U :\ \mathcal{H}_\omega \to \mathcal{H}'_\omega$ defined densely by $U\pi_\omega(a)\Omega_\omega := \pi'_\omega(a)\Omega'_\omega$ is unitary.

Theorem 4.2.
Let ω be a state over a unital $^*-$algebra $\mathfrak{A}$ which is invariant for an element $\alpha \in Aut(\mathfrak{A})$. Then there exists a uniquely determined unitary operator U_ω on the GNS Hilbert space $\mathcal{H}_\omega$ such that

$$U_\omega\ \pi_\omega(a)\ \Omega_\omega = \pi_\omega(\alpha(a))\Omega_\omega \tag{4.2}$$

The proof follows from the uniqueness part of theorem 4.1 applied to the alternative data $(\mathcal{H}_\omega, \pi_\omega \circ \alpha, \Omega_\omega)$.

Corollary 4.1.
Let ω be a $G-$invariant state on a unital $^*-$algebra. Then there is a unitary representation $g \mapsto U_\omega(g)$ of G on the GNS Hilbert space $\mathcal{H}_\omega$ defined by

$$U_\omega(g)\;\pi_\omega(a)\;\Omega_\omega := \pi_\omega(\alpha_g(a))\;\Omega_\omega \tag{4.3}$$

where $g \mapsto \alpha_g$ is the corresponding automorphism group.

Notice that this means that the group G is represented *without anomalies*, that is, there are e.g. no central extensions with non – vanishing obstruction cocycle.

An important concept in connection with a state ω is its *folium*. This is defined as the set of states ω_ρ on $\mathfrak{A}$ defined by

$$\omega_\rho(a) := \frac{\mathrm{Tr}_{\mathcal{H}_\omega}(\rho\pi_\omega(a))}{\mathrm{Tr}_{\mathcal{H}_\omega}(\rho)} \tag{4.4}$$

where ρ is a positive trace class operator on the GNS Hilbert space $\mathcal{H}_\omega$.

If $\mathfrak{A}$ is not only a unital $^*-$algebra but in fact a C^*-algebra then there are many more structural theorems available. For instance one can show, using the Hahn – Banach theorem that representations always exist, that every non – degenerate representation is a direct sum of cyclic representations and that every state is continuous so that the GNS representations are always by bounded operators. While the C^*-norm implies this huge amount of extra structure, a reasonable C^*-norm on a $^*-$algebra is usually very hard to guess unless one actually constructs a representation by bounded operators. We have thus chosen to keep with the more general concept of $^*-$algebras.

In AQFT [4] one uses the mathematical framework of operator algebras, the basics of which we just sketched and combines it with the physical concept of locality of nets of local algebras $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$. That is, given a background spacetime (M, η) consisting of a differentiable $D-$manifold and a background metric η, for each open region $\mathcal{O}$ one assigns a C^*-algebra $\mathfrak{A}(\mathcal{O})$. These are mutually (anti)commuting for spacelike separated (with respect to η) regions. This is the statement of the most important one of the famous Haag – Kastler axioms. The framework is ideally suited to formulate and prove all of the structural theorems of QFT on Minkowski space and even to a large extent on curved spaces [22]. In AQFT one cleanly separates two steps of quantizing a field theory, namely first to define a suitable algebra $\mathfrak{A}$ and then to study its representations in a second step.

In what follows we apply this framework to the bosonic string. However, since the string is a worldsheet background independent theory, we must use a background independent quantization scheme similar to the programme sketched in section 3. In particular, we must extend the framework just presented as to deal with constraints.

5 Elements from Loop Quantum Gravity

LQG is a canonical approach towards quantum gravity based on Dirac's quantization programme for field theories's with constraints. The canonical approach is ideally suited to constructing background metric independent representations of the canonical commutation relations as is needed e.g. in quantum gravity. We will describe this programme in some detail below, again experts can safely skip this section. As we will see, in its modern form Dirac's programme uses some elements of the theory of operator algebras and AQFT but what is different from AQFT is that the canonical approach is, by definition, a quantum theory of the initial data, that is, operator valued distributions are smeared with test functions supported in $(D-1)-$dimensional slices rather than $D-$dimensional regions. This is usually believed to be a bad starting point in AQFT because of the singular behaviour of the $n-$point Wightman distributions of interacting scalar fields in perturbation theory when smeared with "test functions" supported in lower dimensional submanifolds. The

way out of this "no-go theorem" is twofold: 1. In usual perturbation theory one uses very specific (Weyl) algebras and corresponding Fock representations to formulate the canonical commutation relations but the singular behaviour might be different for different algebras and their associated representations. 2. In a reparameterization invariant theory such as General Relativity or string theory all observables are by definition time independent, see e.g. the **Pohlmeyer Charges** constructed above, so that smearing in the additional time direction just multiplies the observables with a constant and thus does not change their singularity structure.

As a pay – off, in contrast to AQFT the framework of LQG does not need a background spacetime as we will see below.

We assume to be given a (possibly infinite dimensional) phase space $\mathcal{M}$ with a Poisson bracket $\{.,.\}$ (technically a strong symplectic structure). Furthermore, we assume to be given a set of first class constraints C_I on $\mathcal{M}$ which we take to be real valued w.l.g. (pass to real and imaginary part if necessary). Here the index set $\mathcal{I}$ in which the indices I take values can be taken to be discrete. This seems not to be the most general situation because e.g. in the case of the string we have the continuous label $x \in S^1$, e.g. for the constraints $C(x)$, however, the constraints always must be smeared with test functions. In fact, let b_I be an orthonormal basis of smooth functions of $L_2(S^1, dx/(2\pi))$ (e.g. $b_I(x) = \exp(iIx),\ I \in \mathbb{Z}$) and define $C_I :=< b_I, C >$. Then $C_I = 0\ \forall I$ is equivalent with $C(x) = 0$ for a.a. $x \in S^1$ and since $C(x)$ is classically a continuous function it follows that $C(x) = 0$ for all $x \in S^1$. That the constraints be first class means that there are structure functions $f_{IJ}{}^K$ on $\mathcal{M}$ (not necessarily independent of $\mathcal{M}$) such that $\{C_I, C_J\} = f_{IJ}{}^K C_K$. We may also have a Hamiltonian H (not a linear combination of the C_I) which is supposed to be gauge invariant, that is, $\{C_I, H\} = 0$ for all I.

Given this set – up, the Canonical Quantization Programme consists, roughly, of the following steps:

I. *Kinematical Poisson Algebra* $\mathfrak{P}$ *and Kinematical Algebra* $\mathfrak{A}$
The first choice to be made is the selection of a suitable Poisson* subalgebra of $C^\infty(\mathcal{M})$. This means that we must identify a subset of functions $f \in C^\infty(\mathcal{M})$ which is closed under complex conjugation and Poisson brackets and which separates the points of $\mathcal{M}$. This choice is guided by gauge invariance, that is, the functions f should have a simple behaviour under the gauge transformations generated by the Hamiltonian vector fields of the C_I on $\mathcal{M}$. At this point it is not important that $\mathfrak{P}$ consists of bounded functions. However, when promoting $\mathfrak{P}$ to an operator algebra, it will be convenient to choose bounded functions of the elements p of $\mathfrak{P}$, say the usual Weyl elements $W = \exp(itp),\ t \in \mathbb{R}$, and to define the algebra of the W to be given by formally imposing the canonical commutation relations among the p, namely that commutators are given by $i\hbar$ times the Poisson bracket and that the operators corresponding to real valued p are self-adjoint. This has the advantage of resulting in bounded operators W which avoids domain questions later on. We will denote the resulting $^*-$algebra generated by the operators W by $\mathfrak{A}$.

II. *Representation Theory of* $\mathfrak{A}$
We study the representation theory of $\mathfrak{A}$, that is, all $^*-$algebra morphisms $\pi:\ \mathfrak{A} \to \mathcal{B}(\mathcal{H}_{Kin})$ where $\mathcal{B}(\mathcal{H}_{Kin})$ denotes the C^*-algebra of bounded operators on a kinematical Hilbert space $\mathcal{H}_{Kin}$. In particular,
$$\pi(\bar{f}) = [\pi(f)]^\dagger,\quad [\pi(f), \pi(f')] = i\hbar\pi(\{f, f'\}) \tag{5.1}$$
and the corresponding (Weyl) relations for the exponentiated elements. Guiding principles here are again gauge invariance and (weak) continuity. Moreover, the representation should be irreducible on physical grounds (otherwise we have superselection sectors implying that the physically relevant information is already captured in a closed subspace). Operator algebra theoretic methods such as the GNS construction are of great importance here.

III. *Selection of a suitable kinematical representation*

Among all possible representations π we are, of course, only interested in those which support the constraints C_I as operators. Since, by assumption, $\mathfrak{A}$ separates the points of $\mathcal{M}$ it is possible to write every C_I as a function of the $f \in \mathfrak{A}$, however, that function is far from unique due to operator ordering ambiguities and in field theory usually involves a limiting procedure (regularization and renormalization). We must make sure that the resulting limiting operators $\pi(C_I)$ are densely defined and closable (i.e. their adjoints are also densely defined) on a suitable domain of $\mathcal{H}_{Kin}$. This step usually severely restricts the abundance of representations. Alternatively, in rare cases it is possible to quantize the finite gauge transformations generated by the classical constraints provided they exponentiate to a group. This is actually what we will do in this paper.

IV. *Solving the Quantum Constraints*

There are essentially two different strategies for solving the constraints, the first one is called "Group Averaging" and the second one is called "Direct Integral Decomposition". The first method makes additional assumtions about the structure of the quantum constraint algebra while the second does not and is therefore of wider applicability.

For "Group Averaging" [23] the first assumption is that the $\pi(C_I)$ are actually self-adjoint operators on $\mathcal{H}_{Kin}$ and that the structure functions $f_{IJ}\ ^K$ are actually constants on $\mathcal{M}$. In this case, under suitable functional analytic assumptions we can define the unitary operators

$$U(t) := \exp(i \sum_I t^I \pi(C_I)) \tag{5.2}$$

where the parameters take range in a subset of $\mathbb{R}$ depending on the $\pi(C_I)$ in such a way that the $U(t)$ define a unitary representation of the Lie group G determined by the Lie algebra generators C_I. In particular, this means that there is no anomaly, i.e. $[\pi(C_I), \pi(C_J)] = i\hbar f_{IJ}\ ^K \pi(C_K)$. The second assumption is that G has an invariant (not necessarily finite) bi − invariant Haar measure μ_H. In this case we may define an anti-linear rigging map

$$\rho:\ \mathcal{H}_{Kin} \to \mathcal{H}_{Phys};\ \psi \mapsto \int_G d\mu_H(t)\ < U(t)\psi, . >_{\mathcal{H}_{Kin}} \tag{5.3}$$

with physical inner product

$$< \rho(\psi), \rho(\psi') >_{\mathcal{H}_{Phys}} := [\rho(\psi')](\psi) \tag{5.4}$$

Notice that $\rho(\psi)$ defines a distribution on (a dense subset of) $\mathcal{H}_{Kin}$ and solves the constraints in the sense that $[\rho(\psi)](U(t)\psi') = [\rho(\psi)](\psi')$ for all $t \in G$. Moreover, given any kinematical algebra element $O \in \mathfrak{A}$ we may define a corresponding Dirac observable by

$$[O] := \int_G d\mu_H(t)\ U(t)\ O\ U(t)^{-1} \tag{5.5}$$

Let us now come to the "Direct Integral Method" [24, 25]. Here we do not need to assume that the $\pi(C_I)$ are self-adjoint. Also the structure functions $f_{IJ}\ ^K$ may have non-trivial dependence on $\mathcal{M}$. In contrast to the string, this is actually the case in 4D General Relativity and for higher p − brane theories which is why only this method is available there. Let us now consider an operator valued positive definite matrix $\hat{Q}^{IJ}$ such that the **Master Constraint Operator**

$$\widehat{\mathbf{M}} := \frac{1}{2} \sum_{I,J} [\pi(C_I)]^\dagger \hat{Q}_{IJ} [\pi(C_J)] \tag{5.6}$$

is densely defined. Obvious candidates for $\hat{Q}_{IJ}$ are quantizations $\pi(Q^{IJ})$ of postive definite $\mathcal{M}-$valued matrices with suitable decay behaviour in $\mathcal{I}-$space. Then, since $\widehat{\mathbf{M}}$ is positive by construction it has self-adjoint extensions (e.g. its Friedrichs extension [16]) and its spectrum is supported on the positive

real line. Let $\lambda_0^{\mathsf{M}} = \inf \sigma(\widehat{\mathsf{M}})$ be the minimum of the spectrum of $\widehat{\mathsf{M}}$ and redefine $\widehat{\mathsf{M}}$ by $\widehat{\mathsf{M}} - \lambda_0^{\mathsf{M}} \mathrm{id}_{\mathcal{H}_{Kin}}$. Notice that $\lambda_0^{\mathsf{M}} < \infty$ by assumption and proportional to $\hbar$ by construction. We now use the well-known fact that $\mathcal{H}_{Kin}$, if separable, can be represented as a direct integral of Hilbert spaces

$$\mathcal{H}_{Kin} \cong \int_{\mathbb{R}^+}^{\oplus} d\mu(\lambda) \mathcal{H}_{Kin}^{\oplus}(\lambda) \tag{5.7}$$

where $\widehat{\mathsf{M}}$ acts on $\mathcal{H}_{Kin}^{\oplus}(\lambda)$ by multiplication by λ. The measure μ and the scalar product on $\mathcal{H}_{Kin}^{\oplus}(\lambda)$ are induced by the scalar product on $\mathcal{H}_{Kin}$. The physical Hilbert space is then simply $\mathcal{H}_{Phys} := \mathcal{H}_{Kin}^{\oplus}(0)$ and Dirac observables are now constructed from bounded self-adjoint operators O on $\mathcal{H}_{Kin}$ by the *ergodic mean*

$$[O] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} dt\; e^{it\widehat{\mathsf{M}}}\; O\; e^{-it\widehat{\mathsf{M}}} \tag{5.8}$$

and they induce bounded self-adjoint operators on $\mathcal{H}_{Phys}$.

Notice that both methods can be combined. Indeed, it may happen that a subset of the constraints can be solved by group averaging methods while the remainder can only be solved by direct integral decomposition methods. In this case, one will construct an intermdediate Hilbert space which the first set of constraints annihilates and which carries a representation of the second set of constraints. This is actually the procedure followed in LQG and it will be convenient to do adopt this "solution in two steps" for the string as well in our companion paper [13] for curved target spaces. For the purposes of this paper, group averaging will be sufficient.

V. *Classical Limit*

Notice that our construction is entirely non-perturbative, there are no (at least not necessarily) Fock spaces and there is no perturbative expansion (Feynman diagrammes) even if the theory is interacting. While this is attractive, the prize to pay is that the representation $\mathcal{H}_{Kin}$ to begin with and also the final physical Hilbert space $\mathcal{H}_{Phys}$ will in general be far removed from any physical intuition. Hence, we must make sure that what we have constructed is not just some mathematical object but has, at the very least, the classical theory as its classical limit. In particular, if classical Dirac observables are known, then the quantum Dirac observables (5.8) and (5.8) should reduce to them in the classical limit. To address such questions one must develop suitable semiclassical tools.

We see that the construction of the quantum field theory in AQFT as well as in LQG is nicely separated: First one constructs the algebra and then its representations. What is new in LQG is that it also provides a framework for dealing with constraints. Now LQG is more than just offering a framework, this framework has been carried out systematically with quite some success, so far until step IV, see e.g. [2] for exhaustive reports. That is, not only did one pick a suitable kinematical algebra $\mathfrak{A}$ but also constructed a non-trivial representation thereof by use of the GNS construction which moreover supports the quantum constraints. In what follows we will apply this programme systematically to the closed, bosonic string on flat backgrounds. Due to the flatness of the backgorund and the fact that the constraints close with structure constants rather than structure functions we can carry out our programme rather effortlessly in a novel representation that is not available in LQG. In our companion paper [13] we will stick closer to the kind of representations used in LQG.

6 The Closed, Bosonic LQG – String on Flat Target Spaces

We can now finally pick up the fruits of our efforts and combine sections three through five to construct the LQG – String. We will do this by systematically following the canonical quantization programme.

6.1 General Representation Problem

Before we go into details, let us state the general representation problem:

In view of the the theory of invariants of section 3 we take as our classical Poisson $^*-$subalgebra $\mathfrak{P}$ the algebra of the $Y_\pm(f)$ of section 3. By varying the smooth smearing functions f_μ we can extract $\pi_\mu(x)$ as well as $X^{\mu\prime}(x)$ for all $x \in S^1$. Only a global constant X_0^μ remains undetermined. Furthermore, we include into $\mathfrak{P}$ the Lorentz generators $J_{\mu\nu}$ which allows us to extract X_0^μ up to the the translation freedom $X_0^\mu \to t\ell_s^2 p^\mu,\ t \in \mathbb{R}$. This is the only information which we cannot extract from $\mathfrak{P}$ which is admissable in view of the fact that the physical invariants of the string do not depend on it. In this sense, $\mathfrak{P}$ seperates the points of the unconstrained phase space $\mathcal{M}$.

Next we construct from $\mathfrak{P}$ bounded functions on $\mathcal{M}$ which still separate the points and promote them to operators by asking that Poisson brackets and complex conjugation on $\mathfrak{P}$ be promoted to commutators divided by $i\hbar$ and the adjoint respectively. Denote the resulting $^*-$algebra by $\mathfrak{A}$.

The automorphism groups $\alpha^\pm_\varphi,\ \varphi \in \mathrm{Diff}(S^1)$ generated by the Virasoro constraints as well as the Poincaré automorphism group $\alpha_{a,L}$ extend naturally from $\mathfrak{P}$ to $\mathfrak{A}$ simply by $\alpha_.(W(Y_\pm)) = W(\alpha_.(Y_\pm))$. A general representation of $\mathfrak{A}$ should now be such that the automorphism groups $\alpha_.$ are represented by inner automorphisms, that is, by conjugation by unitary operators representating the corresponding group elements. Physically the representation property amounts to an anomaly – free implementation of both the local gauge group and the global symmetry group while unitarity implies that expectation values of gauge invariant or Poincaré invariant observables does not depend on the gauge or frame of the measuring state. Finally, the representation should be irreducible or at least cyclic.

As we have sketched in section 4, a powerful tool to arrive at such representations is via the GNS construction. Hence we will be looking at representations that arise from a positive linear functional ω on $\mathfrak{A}$ which is invariant under all the automorphisms $\alpha_.$. Since $\mathfrak{A}$ is not (naturally) a C^*-algebra, this is not the most general representation but it is important subclass thereof.

These representations are still kinematical, that is, the corresponding GNS Hilbert space $\mathcal{H}_{Kin} := \mathcal{H}_\omega$ still represents $\mathfrak{A}$ and not only physical observables. To arrive at the physical Hilbert space $\mathcal{H}_{Phys}$ we make use of the group averaging method outlined in section 5:
Let $\Phi_{Kin} := \Phi_\omega$ be the dense subspace of $\mathcal{H}_\omega$ defined by the $\pi_\omega(a)\Omega_\omega,\ a \in \mathfrak{A}$. Let Φ_ω^* be the algebraic dual of Φ_ω (linear functionals without continuity requirements). Suppose that μ is a translation bi – invariant measure on $\mathrm{Diff}(S^1)$ such that the following anti – linear map exists

$$\eta_\omega:\ \Phi_\omega \to \Phi_\omega^*;\ \pi_\omega(a)\Omega_\omega \mapsto \int_{[\mathrm{Diff}(S^1)]^2} d\mu(\varphi^+)\ d\mu(\varphi^-)\ < U_\omega^+(\varphi^+)U_\omega^-(\varphi^-)\pi_\omega(a)\Omega_\omega, . >_{\mathcal{H}_\omega} \tag{6.1}$$

where we have used that the U_ω^+ commute with the U_ω^-. Consider the sesqui – linear form

$$\begin{aligned} & < \eta_\omega(\pi_\omega(a)\Omega_\omega), \eta_\omega(\pi_\omega(b)\Omega_\omega) >_{Phys} \\ := & [\eta_\omega(\pi_\omega(b)\Omega_\omega)](\pi_\omega(a)\Omega_\omega) \\ = & \int_{[\mathrm{Diff}(S^1)]^2} d\mu(\varphi^+)\ d\mu(\varphi^-)\ < U_\omega^+(\varphi^+)U_\omega^-(\varphi^-)\pi_\omega(b)\Omega_\omega, \pi_\omega(a)\Omega_\omega >_{\mathcal{H}_\omega} \end{aligned} \tag{6.2}$$

Fortunately, the steps to rigorously define the rigging map (6.1) and the physical inner product (6.2) have been completed for a general theory already in [26] so that we can simply copy those results.

This concludes the definition of our general representation problem. We now pass to a specific choice for the kinematical algebra.

6.2 Concrete Implementation

We now propose a particular choice of $\mathfrak{A}$ based on our experiences with background independent representations that proved useful in LQG [2]. Then, rather than studying its general representation theory, we confine

ourselves in this paper to constructing a specific positive linear functional and show that the **Pohlmeyer Charges** can be defined on the corresponding physical Hilbert space in the sense of (6.2). This then proves that the programme outlined in section 6.1 has non – trivial solutions.

Let I denote Borel subsets of S^1, that is, sets generated by countable intersections and unions from closed subsets of $[0, 2\pi)$ (with periodic identifications). We consider instead of general smearing functions $f_\mu(x)$ the more specific ones

$$f_\mu^{I,k}(x) = k_\mu \chi_I(x) \tag{6.3}$$

where k_μ are real valued numbers with dimension cm^{-1} and χ_I denotes the characteristic function of I. The corresponding smeared $Y_\epsilon^k(I) := Y_\epsilon(f^{I,k})$, $\epsilon = \pm 1$ satisfy the Poisson algebra

$$\{Y_\epsilon^k(I), Y_{\epsilon'}^{k'}(I')\} = \epsilon\alpha'\delta_{\epsilon,\epsilon'}\eta^{\mu\nu}k_\mu k'_\nu\{[\chi_I]_{\partial I'} - [\chi_{I'}]_{\partial I}\} =: \alpha'\alpha(\epsilon, k, I; \epsilon, k', I') \tag{6.4}$$

where the boundary points in ∂I of I are signed according to whether they are right or left boundaries of closed intervals. The functionals $Y_\epsilon^k(I)$, together with the $J_{\mu\nu}$, evidently still separate the points of $\mathcal{M}$ and form a closed Poisson $^*-$subalgebra.

The kinematical quantum algebra will be generated from the *Weyl elements*

$$\hat{W}_\epsilon^k(I) := e^{i\hat{Y}_\epsilon^k(I)} \tag{6.5}$$

for which we require canonical commutation relations induced from

$$[\hat{Y}_\epsilon^k(I), \hat{Y}_{\epsilon'}^{k'}(I')] = i\hbar\{Y_\epsilon^k\widehat{(I), Y_{\epsilon'}^{k'}}(I')\} \tag{6.6}$$

It follows from the Baker – Campbell – Hausdorff formula that

$$W_\epsilon^k(I)W_{\epsilon'}^{k'}(I') = \exp(-i\ell_s^2\alpha(\epsilon, k, I; \epsilon, k', I')/2)\ \exp(i[Y_\epsilon^k(I) + Y_{\epsilon'}^{k'}(I')]) \tag{6.7}$$

in particular, Weyl elements with $\epsilon \neq \epsilon'$ are commuting. Since

$$Y_\pm^k(I) + Y_\pm^{k'}(I') = Y_\pm^{k+k'}(I \cap I') + Y_\pm^k(I - I') + Y_\pm^{k'}(I' - I) \tag{6.8}$$

we see that a general element of $\mathfrak{A}$ can be written as a finite, complex linear combinations of elements of the form

$$W_+^{k_1..k_M}(I_1, .., I_M)W_-^{l_1..l_N}(J_1, .., J_N) \tag{6.9}$$

where

$$W_\pm^{k_1,..,k_M}(I_1, .., I_M) = \exp(i[\sum_{m=1}^M Y_\pm^{k_m}(I_m)]) \tag{6.10}$$

where all the I_m are non – empty and mutually non – overlapping (they intersect at most in boundary points). These considerations motivate the following definition.

Definition 6.1.
i)
A momentum network $s = (\gamma(s), k(s))$ is a pair consisting of a finite collection $\gamma(s)$ of non – empty and non – overlapping (up to boundary points) closed, oriented intervals I of S^1, together with an assignment $k(s)$ of momenta $k_\mu^I(s)$ for each interval. We will use the notation $I \in \gamma$ if I is an element of the collection of intervals γ to which we also refer as edges.
ii)
A momentum network operator with parity ϵ is given by

$$W_\epsilon(s) = \exp(i[\sum_{I\in\gamma(s)} Y_\epsilon^{k^I(s)}(I)]) \tag{6.11}$$

This definition has a precise counterpart in LQG in terms of spin networks and spin network (multiplication) operators where the intervals are replaced by oriented edges of a graph which is the counterpart of the collection of intervals here. The spin quantum numbers of LQG are replaced by the momentum labels of the string.

The general *Weyl Relations* are then given by

$$\begin{aligned} W_+(s_1)\, W_-(s_1')W_+(s_2)W_-(s_2') &= e^{-\frac{i}{2}\ell_s^2[\alpha(s_1,s_2)-\alpha(s_1',s_2')]}\, W_+(s_1+s_2)\, W_-(s_1'+s_2') \\ [W_+(s_1)\, W_-(s_2)]^* &= W_+(-s_1)\, W_-(-s_2) \end{aligned} \tag{6.12}$$

where

$$\alpha(s_1,s_2)=\sum_{I_1\in\gamma(s_1)}\ \sum_{I_2\in\gamma(s_2)} [\eta^{\mu\nu}k^{I_1}_\mu(s_1)k^{I_1}_\mu(s_1)]\alpha(I_1,I_2) \tag{6.13}$$

with $\alpha(I,J)=[\chi_I]_{|\partial J}-[\chi_J]_{|\partial J}$. The notation s_1+s_2 means that we decompose all the $I_1\in\gamma(s_1)$ and $I_2\in\gamma(s_2)$ into their unique, maximal, mutually non – overlapping segments and assign the momentum $k^{I_1}(s_1)+K^{I_2}(s_2)$ to the segment $I_1\cap I_2$ and $k^{I_1}(s_1)$, $k^{I_2}(s_2)$ respectively to the segments $I_1-\gamma(s_2)$ and $I_2-\gamma(s_1)$ respectively. Likewise $-s$ is the same as s just that $I\in\gamma(s)$ is assigned $-k^I(s)$. See (6.8) for an example. The relations (6.12) are the direct analog of the holonomy – flux Weyl algebra underlying LQG [27]. Notice that the "holonomies" along intervals I

$$h^k_I(X):=\exp(2ik_\mu\int_I dx X^{\mu\prime})=W^k_+(I)W^{-k}_-(I) \tag{6.14}$$

are mutually commuting due to the difference in the sign of the phase factor in (6.12). The holonomies define a maximal Abelean subalgebra in $\mathfrak{A}$. The same is true for the exponentiated "fluxes"

$$F^k_I(\pi):=\exp(2ik^\mu\int_I dx\pi_\mu)=W^k_+(I)W^k_-(I) \tag{6.15}$$

We can combine the local gauge group generated by the Virasoro constraints and the global Poincaré group to the total group

$$G:=\mathrm{Diff}(S^1)\times\mathrm{Diff}(S^1)\times\mathcal{P} \tag{6.16}$$

given by the direct product of two copies of the diffeomorphism group of the circle and the Poincaré group which itself is the semi – direct product of the Lorentz group with the translation group. It has the following action on our Weyl algebra $\mathfrak{A}$ by the automorphisms derived in section 3

$$\alpha_{\varphi_+,\varphi_-,(L,a)}(W_+(s_1)\,W_-(s_2))=\alpha_{\varphi_+,\mathrm{id},(L,0)}(W_+(s_1))\,\alpha_{\mathrm{id},\varphi_-,(L,0)}(W_-(s_2))=W_+([\varphi_+,L]\cdot s_1)\,W_-([\varphi_-,L]\cdot s_2)) \tag{6.17}$$

where

$$\begin{aligned} (\varphi,L)\cdot s &= (\gamma((\varphi,L)\cdot s),\{k^I((\varphi,L)\cdot s)\}_{I\in\gamma((\varphi,L)\cdot s)}) \\ \gamma((\varphi,L)\cdot s) &:= \varphi(\gamma(s)) \\ k^{\varphi(I)}((\varphi,L)\cdot s) &:= L\cdot k^I(s),\ I\in\gamma(s) \end{aligned} \tag{6.18}$$

In other words, the diffeomorphism maps the intervals to their diffeomorphic image but leaves the momenta unchanged while the Lorentz transformation acts only on the momenta. The translations have no effect on the Weyl elements because they only depend on X^μ through the derivatives X'.

In order to find representations of $\mathfrak{A}$ via the GNS construction we notice that

$$\mathfrak{A}=\mathfrak{A}_+\ \otimes\mathfrak{A}_- \tag{6.19}$$

is the tensor product of two Weyl algebras which are isomorphic up to the sign difference in the phase in (6.12). Now we can use the lemma that if $\omega_\pm$ is a positive linear functional on the $^*-$algebra $\mathfrak{A}_\pm$ then $\omega:=\omega_+\otimes\omega_-$ defined by $\omega(a_+\otimes a_-):=\omega_+(a_+)\omega_-(a_-)$ is a positive linear functional on $\mathfrak{A}_+\otimes\mathfrak{A}_-$, see e.g. [28] for a simple proof. Hence it suffices to specify $\omega_\pm$ separately.

6.3 A Specific Example

We should now enter the general representation theory of $\mathfrak{A}_\pm$ or at least the subclass of cyclic and $G-$invariant representations that arise via the GNS construction. Since we have not made any continuity assumptions about the representation of the one parameter unitary groups $t \mapsto W_\pm(ts)$ (where ts is the same as s just that the momenta are rescaled by t) whithout which even the famous Stone – von Neumann uniqueness theorem for the representation theory of the Weyl algebra underlying simple quantum mechanics fails [6, 29], we expect this problem to be rather complex and we leave it as an important project for further research. In this paper we will content ourselves with giving just one non – trivial example. Here it is:

$$\omega_\pm(W_\pm(s)) := \delta_{s,0} \tag{6.20}$$

where $s = 0$ denotes the trivial momentum network consisting of no intervals and zero momenta. This functional is tailored after the positive linear functional underlying the Ashtekar – Lewandowski representation for LQG [30] which only recently [27] has been shown to be the unique (cyclic, spatially diffeomorphism invariant and in part weakly continuous) representation of the LQG Weyl algebra. Expression (6.20) looks rather trivial at first sight but the rest of the paper is devoted to demonstrating that it contains some interesting structure.

Let us first check that it is indeed a $G-$invariant positive linear functional. First of all

$$\omega_\pm(\alpha_g(W_\pm(s))) = \omega_\pm(W_\pm(g\cdot s)) = \delta_{g\cdot s,0} = \delta_{s,0} = \omega_\pm(W_\pm(s)) \tag{6.21}$$

since the $G-$action on the momentum networks preserves the cases $s = 0$ and $s \neq 0$. It follows that we obtain a unitary representation of G defined by

$$U_\omega(g)\pi_\omega(W_+(s)W_-(s))\Omega_\omega = \pi_\omega(W_+(g\cdot s)W_-(g\cdot s))\Omega_\omega \tag{6.22}$$

Next consider a generic element of $\mathfrak{A}_\pm$ given by

$$a_\pm = \sum_{n=1}^N z_n W_\pm(s_n) \tag{6.23}$$

where the s_n are mutually different. Then

$$\begin{aligned}
\omega_\pm((a_\pm)^* a_\pm) &= \sum_{m,n} \bar{z}_m\; z_n\; \omega_\pm([W_\pm(s_m)]^*\; W_\pm(s_n)) \\
&= \sum_{m,n} \bar{z}_m\; z_n\; \omega_\pm(W_\pm(-s_m)\; W_\pm(s_n)) \\
&= \sum_{m,n} \bar{z}_m\; z_n\; e^{\pm\frac{i}{2}\ell_s^2\alpha(s_m,s_n)}\; \omega_\pm(W_\pm(s_n-s_m)) \\
&= \sum_{m,n} \bar{z}_m\; z_n\; e^{\pm\frac{i}{2}\ell_s^2\alpha(s_m,s_n)}\; \delta_{s_n-s_m,0} \\
&= \sum_{n=1}^N |z_n|^2
\end{aligned} \tag{6.24}$$

since $\alpha(s,s) = 0$. Thus $\omega_\pm([a_\pm]^*\; a_\pm) \geq 0$ and equality occurs only for $a_\pm = 0$. Hence the functional is definite and there is no null ideal to be divided out, see section 4. Thus the GNS Hilbert space $\mathcal{H}_{\omega_\pm}$ is simply the Cauchy completion of the algebra $\mathfrak{A}_\pm$ considered as a vector space with cyclic vector $\Omega_{\omega_\pm} = \mathbf{1} = W_\pm(0)$ and representation $\pi_{\omega_\pm}(a_\pm) = a_\pm$. The GNS Hilbert space for $\mathfrak{A}$ is of course the tensor product $\mathcal{H}_\omega = \mathcal{H}_{\omega_+}\otimes\mathcal{H}_{\omega_-}$.

One might wonder whether this representation is unitarily equivalent to the Ashtekar – Isham – Lewandowski representation [30] for the string, or rather its direct analog for scalar fields [31, 14, 32] which we are going

to consider in great detail in the companion paper [13]. If that were the case then we could identify Ω_ω with the constant state $\mathbf{1}$, the operators $A(s) := \pi_\omega(W_+(s/2))\pi_\omega(W_-(s/2))^{-1}$ would be multiplication "holonomy" operators on $\mathcal{H}_\omega = L_2(\overline{\mathcal{A}}, d\mu_0)$ where $\overline{\mathcal{A}}$ is a certain space of distributional "connections" $X^{\mu\prime}$ and the operators $\exp(i\pi(s)) := \pi_\omega(W_+(s/2))\pi_\omega(W_-(s/2))$ would be exponentials of derivative operators with $\exp(i\pi(s))\Omega_\omega = \Omega_\omega$. But this is not the case since $\omega(\exp(i\pi(s)) = \delta_{s,0}$ while clearly $\omega_{AIL}(\exp(i\pi(s))) = 1$. In the Ashtekar – Isham – Lewandowski representation the derivative operators exist (the corresponding one parameter unitary Weyl groups are weakly continuous) while in our representation the derivative operators do not exist (the Weyl groups are discontinuous). Thus the corresponding GNS representations are unitarily inequivalent.

6.4 Mass Spectrum

According to (6.22) all states are translation invariant if we define a unitary representation of G, as in section 4, by

$$U_\omega(g)\pi_\omega(b)\Omega_\omega = \pi_\omega(\alpha_g(b))\Omega_\omega \tag{6.25}$$

for all $b \in \mathfrak{A}$ and since, as we will see, all invariant charges are constructed from the $\pi_\omega(W_\pm(s))$ by taking certain limits, also states created by the charges will be translation invariant. Hence the total momentum and hence the mass of our states vanishes identically in drastic contrast with usual string theory. We now generalize our construction as to include *arbitrary non – negative mass spectrum (tachyon – freeness)*.

We introduce a D – parameter family of states $p \mapsto \omega_p$ on $\mathfrak{A}$ which as far as $\mathfrak{A}$ is concerned are just copies of our $\omega := \omega_0$ constructed in the previous section. However, we generalize (6.25) to

$$V_\mu(g)\pi_{\omega_p}(b)\Omega_{\omega_p} = e^{ia\cdot p}\pi_{\omega_{L^T p}}(\alpha_g(b))\Omega_{\omega_{L^T p}} \tag{6.26}$$

where L^T is the transpose of the Lorentz transformation datum in $g := (\varphi_+, \varphi_-, L, a)$ and $(\mathcal{H}_{\omega_p}, \pi_{\omega_p}, \Omega_{\omega_p})$ are the GNS data associated with ω_p. Of course, all these GNS data are mutually unitarily equivalent. It is easy to check that (6.26) satisfies the representation property.

Let $\overline{V}_+$ be the closure of the interior of the future lightcone and let μ be a quasi – $\mathfrak{L}_0^+$ – invariant probability measure on $\mathbb{R}^D$ with support in $\overline{V}_+$. Here $\mathfrak{L}_0^+$ denotes the connected component of $SO(1, D-1)$ which preserves the sign of p^0 (proper orthochronous Lorentz group). Recall that a measure μ on a measurable space $\mathfrak{X}$ is said to be quasi – invariant for a group G acting on $\mathfrak{X}$ if μ and all its translates have the same measure zero sets. In other words, if E is a measurable set of $\mathfrak{X}$ and $g \cdot E$ is the translation of E by $g \in G$ then $\mu(E) = 0$ implies $\mu_g(E) := \mu(g \cdot E) = 0$ for all $g \in G$. This condition implies that the translated measures are mutually absolutely continuous and that the Radon – Nikodym derivative $d\mu_p/d\mu$ is a well – defined, positive $L_1(\mathfrak{X}, d\mu)$ function. The choice of μ is not very important for the theory of induced representations as sketched below in the sense that mutally absolutely continuous measures lead to unitarily equivalent representations. See e.g. [33] for all the details.

We now construct the direct integral of the Hilbert spaces $\mathcal{H}_{\omega_p}$, that is,

$$\mathcal{H}_\mu := \int_{\mathbb{R}^D}^{\oplus} d\mu(p)\, \mathcal{H}_{\omega_p} \tag{6.27}$$

The definition of (6.27) is as follows: Let $\mathfrak{X}$ be a locally compact space, μ a measure on $\mathfrak{X}$ and $x \mapsto \mathcal{H}_x$ an assignment of Hilbert spaces such the function $x \mapsto n_x$, where n_x is the dimension of of $\mathcal{H}_x$, is measurable. It follows that the sets $\mathfrak{X}_n = \{x \in \mathfrak{X};\ n_x = n\}$, where n denotes any cardinality, are measurable. Since Hilbert spaces whose dimensions have the same cardinality are unitarily equivalent we may identify all the $\mathcal{H}_x,\ n_x = n$ with a single $\mathcal{H}_n$. We now consider maps

$$\xi :\ \mathfrak{X} \to \prod_{x \in \mathfrak{X}} \mathcal{H}_x;\ \ x \mapsto (\xi(x))_{x \in \mathfrak{X}} \tag{6.28}$$

subject to the following two constraints:
1. The maps $x \mapsto < \xi, \xi(x) >_{\mathcal{H}_n}$ are measurable for all $x \in \mathfrak{X}_n$ and all $\xi \in \mathcal{H}_n$.
2. If

$$< \xi_1, \xi_2 > := \sum_n \int_{\mathfrak{X}_n} d\mu(x) < \xi_1(x), \xi_2(x) >_{\mathcal{H}_n} \tag{6.29}$$

then $< \xi, \xi > < \infty$.
The completion of the space of maps (6.28) in the inner product (6.29) is called the direct integral of the $\mathcal{H}_x$ with respect to μ and one writes

$$\mathcal{H}_\mu = \int_{\mathfrak{X}}^{\oplus} d\mu(x)\ \mathcal{H}_x, \quad < \xi_1, \xi_2 > = \int_{\mathfrak{X}} d\mu(x)\ < \xi_1(x), \xi_2(x) >_{\mathcal{H}_x} \tag{6.30}$$

See [21] for more details.

Since in our case the dimension of all the $\mathcal{H}_{\omega_p},\ p \in \mathbb{R}^D =: \mathfrak{X}$ has the same cardinality, the measurability condition on the dimension function is trivially satisfied. In our case the $\xi(p),\ p \in \mathbb{R}^D$ are of the form

$$\xi(p) = \pi_{\omega_p}(b_p)\Omega_{\omega_p} \tag{6.31}$$

and provided the assignment $\mathbb{R}^D \to \mathfrak{A};\ p \mapsto b_p$ satisfies the covariance condition $\alpha_g(b_p) = \alpha_{\varphi_+,\varphi_-}(b_{L^T p})$ we obtain a *unitary* representation of G on $\mathcal{H}_\mu$ by

$$[U_\mu(g)\xi](p) := \sqrt{\frac{d\mu_L(p)}{d\mu(p)}}[V_\mu(g)\xi](p) \tag{6.32}$$

where V_μ was defined in (6.26). The construction (6.32) is, of course, standard in QFT and is related to the *induced representation* of G by the little groups of the subgroup $H =$Diff$_+(S^1)\times$Diff$_-(S^1) \times \mathfrak{L}_0^+$. See [34] for more details.

A particularly nice set of states are the "diagonal states" which arise from the GNS data corresponding to the state

$$\omega_\mu := \int_{\mathbb{R}^D} d\mu(p)\ \omega_p \tag{6.33}$$

which is just the convex linear combination of the ω_p so that

$$\Omega_{\omega_\mu} = (\Omega_{\omega_p})_{p\in\mathbb{R}^D},\ \ \pi_{\omega_\mu}(b)\Omega_{\omega_\mu} = (\pi_{\omega_p}(b)\Omega_{\omega_p})_{p\in\mathbb{R}^D} \tag{6.34}$$

hence $b_p = b$ does not depend on p and the measurability condition is trivially satisfied. We clearly have $\omega_\mu(b) = \omega_0(b)$ so the diagonal states are normalizable if and only if $\omega_0(b^* b) < \infty$. More generally we consider "almost diagonal" states of the form $(f(p)\pi_{\omega_p}(b)\Omega_{\omega_p})_{p\in\mathbb{R}^D})$ where f is any $L_2(\mathbb{R}^D, d\mu)$ function. They obviously define a closed invariant subspace of $\mathcal{H}_\mu$. They do not satisfy the covariance condition because α_L does not act on f, however, since $\omega_p \circ \alpha_L = \omega_0$ for all p this turns out to be sufficient to guarantee unitarity, see below.

We can now characterize an important subclass of quasi – $\mathfrak{L}_0^+$ – invariant probability measures with support in $\overline{V}_+$ a bit closer. Namely, let μ_0 be an actually $\mathfrak{L}_0^+$ – invariant measure and consider a quasi – invariant measure of the form $\mu = |f|^2\mu_0$ where f is a normalized element of $L_2(\mathbb{R}^D, d\mu_0)$ which is μ_0 a.e. non – vanishing. For instance, f could be smooth and of rapid decrease. It follows from the proof of the *Källen – Lehmann* representation theorem for the two – point function of an interacting Wightman scalar field that a polynomially bounded μ_0 is necessarily of the form

$$d\mu_0(p) = c\delta(p) + d\rho(m)d\nu_m(\vec{p}) \tag{6.35}$$

where $c \geq 0,\ d\nu_m(\vec{p}) = d^{D-1}p/\sqrt{m^2 + \vec{p}^2}$ is the standard $\mathfrak{L}_0^+$ – invariant measure on the positive mass hyperboloid $H_m = \{p \in \mathbb{R}^D;\ p \cdot p = -m^2 \leq 0,\ p^0 \geq 0\}$ and $d\rho(m)$ is a polynomially bounded measure on

$[0,\infty)$, sometimes called the *Källen – Lehmann* spectral measure because it characterizes the mass spectrum of a Wightman field. See [35] for an instructive proof. Hence the freedom in μ_0 boils down to c, ρ.

The point of all these efforts is of course that the translation group of G, in contrast to the rest of G, is represented weakly continously on $\mathcal{H}_\mu$ because it acts trivially on $b \in \mathfrak{A}$. The momentum (generalized) eigenstates are precisely the $\xi(p)$ in (6.31), that is, we can define a self – adjoint operator $\pi_\mu(p_\nu)$ as the generator of the translation subgroupf of G by $[\pi_\mu(p_\nu)\xi](p) = p_\nu \xi(p)$. Notice that all the states of $\mathcal{H}_{\omega_p}$ have the same mass $m^2 = -p \cdot p \geq 0$, hence *there is no tachyon*.

There is a different way to look at the almost diagonal states: Define

$$\xi(p) := f(p)\pi_{\omega_p}(b)\Omega_{\omega_p}, \quad f \in L_2(\mathbb{R}^D, d\mu_0) \tag{6.36}$$

and $\mu := \mu_0$ so that the Jacobean in (6.32) equals unity. Then

$$\begin{aligned} ||U_{\mu_0}(g)\xi||^2 &= \int_{\mathbb{R}^D} d\mu_0(p)\; \omega_{L^T p}([f(p)\alpha_g(b)]^*\; [f(p)\alpha_g(b)]) \\ &= \int_{\mathbb{R}^D} d\mu_0(p)\; |f(p)|^2 \omega_{L^T p}(\alpha_g(b^* b)) \\ &= ||f||_2^2\; \omega_0(b^* b) = ||\xi||^2 \end{aligned} \tag{6.37}$$

reproving unitarity. In what follows we focus, for simplicity, on the closed subspace of $\mathcal{H}_\mu$ defined by the almost diagonal states (6.36) which in turn is completely characterized by $\mathcal{H}_\omega$ as far as $\mathfrak{A}$ is concerned but includes the additional twist with respect to the representation of G displayed in this section which enables us to add massive states to the theory. For instance we could have

$$\rho(m) = \sum_{n=0}^{\infty} \delta(m, n/\ell_s) \tag{6.38}$$

corresponding to the usual string theory mass spectrum with the tachyon removed. Keeping this in mind, it will suffice to consider $\mathcal{H}_\omega$ in what follows (equivalent to setting $c = 1, \rho = 0$).

6.5 Implementation of the Pohlmeyer Charges

We saw at the end of section 6.3 that the one – parameter unitary groups $t \mapsto \pi_\omega(W_\pm(ts))$ are not weakly weakly continuous on $\mathcal{H}_\omega$. Since the **Pohlmeyer Charges** $Z_\pm$ involve polynomials of the $Y_\pm$ rather than polynomials of the $W_\pm$ it seems that our representation does not support the **Quantum Pohlmeyer Charges**. Indeed, one would need to use derivatives with respect to t at $t = 0$ to "bring down" the $Y^k_\pm(I)$ from the exponent. In fact, notice the classical identity for the **Pohlmeyer Charges**

$$R^{\mu_1..\mu_n}_\pm(x) = \frac{1}{i^n} \int_{x \leq x_1 \leq ..x_n \leq x+2\pi} d^n x\; [\frac{\delta^n}{\delta f_{\mu_1}(x_1)..\delta f_{\mu_n}(x_n)}]_{|k_1=..=k_n=0}\; W_\pm(f_1)..W_\pm(f_n) \tag{6.39}$$

where $W_\pm(f) = \exp(iY_\pm(f))$. While it is indeed possible to extend ω to the $W_\pm(f)$ by $\omega(W_\pm(f)) := \delta_{f,0}$ (notice that this is a Kronecker δ, not a functional $\delta-$distribution) the functional derivatives in (6.24) are clearly ill – defined when trying to extend ω to the invariants by

$$\begin{aligned} \omega(R^{\mu_1..\mu_n}_\pm(x)) \;\; := \;\; & \frac{1}{i^n} \int_{x \leq x_1 \leq ..x_n \leq x+2\pi} d^n x\; \frac{1}{n!} \sum_{\pi \in S_n} \times \\ & \times [\frac{\delta^n}{\delta f_{\mu_1}(x_1)..\delta f_{\mu_n}(x_n)}]_{|k_1=..=k_n=0}\; \omega(W_\pm(f_{\pi(1)})..W_\pm(f_{\pi(n)})) \end{aligned} \tag{6.40}$$

where we have introduced a symmetric ordering of the the non – commuting $W_\pm(f)$. Expression (6.40) is the direct analog for how to define $n-$point Schwinger functions in Euclidean field theory from the generating

(positive) functional of a probability measure. This works there because the Osterwalder – Schrader axioms [28] require the generating functional to be analytic in f. This is clearly not the case for our ω.

In fact, it is quite hard to construct positive linear, $G-$invariant functionals ω which are weakly continuous or more regular than the one we have found above. The reason for this is the absence of any (worldsheet) background metric in our worldsheet background independent theory. The analytic positive linear functionals for free ordinary Quantum Field Theories all make strong use of a spatially Euclidean background metric which we do not have at our diposal here.

Another difficulty is the Minkowski signature of the flat target space metric. The simplest more regular ansatz will try to invoke the positivity theorems of Gaussian measures from Euclidean QFT, however, these theorems are not applicble due to the Minkowski signature. If we would change to Euclidean target space signature then we could still construct **Pohlmeyer Charges**, however, now they would be complex valued because they and the the Virasoro generators would be based on the complex objects $Y_\pm = \pi \pm iX'$. This would imply that the Weyl elements are no longer unitary, rather we would get something like $(W_\pm(f))^* = W_\mp(-f)$, and in particular the Weyl relations would pick up not only a phase but actually some unbounded positive factor which makes the construction of a positive linear functional even harder. In any case, it would be unclear how to relate this "Euclidean string" to the actual Minkowski string because our Weyl algebra is non – commutative so that the usual Wick rotation is not well-defined.

In conclusion, while we certainly have not yet analyzed the issue systematically enough, (6.20) is the only solution to our representation problem that we could find so far and we must now try to implement the quantum invariants by using a suitable regularization. This means that we write the regulated invariants as polynomials in the $W_\pm(s)$ and then remove the regulator and see whether the result is well – defined and meaningful.

The first step is to consider instead of the invariants $Z_\pm^{\mu_1..\mu_n}$ the functions

$$Z_\pm^{k_1..k_n} := k_{\mu_1} \, .. \, k_{\mu_n} \; Z_\pm^{\mu_1..\mu_n} \tag{6.41}$$

from which the original invariants are regained by specializing the $k_1, .., k_n$. Notice that the $Z^{k_1..k_n}$, in contrast to the $Z^{\mu_1..\mu_n}$, are dimensionless.

Using the fact that classically

$$\frac{W_\pm^k([a,b]) - W_\pm^{-k}([a,b])}{2i[b-a]} = k_\mu Y_\pm^\mu(\frac{a+b}{2}) + O((b-a)^2) \tag{6.42}$$

we can write $Z_\pm^{k_1..k_n}$ as the limit of a Riemann sum

$$\begin{aligned}
&\pi_\omega(Z_\pm^{k_1..k_n}) = \lim_{\mathcal{P}\to S^1} \pi_\omega(Z_{\pm,\mathcal{P}}^{k_1..k_n}) \\
&\pi_\omega(Z_{\pm,\mathcal{P}}^{k_1..k_n}) = \pi_\omega(R_{\pm,\mathcal{P}}^{k_1..k_n}(0)) + \pi_\omega(R_{\pm,\mathcal{P}}^{k_2..k_n k_1}(0)) + .. + \pi_\omega(R_{\pm,\mathcal{P}}^{k_n k_1..k_{n-1}}(0)) \\
&\pi_\omega(R_{\pm,\mathcal{P}}^{k_1..k_n}(0)) = \sum_{m_1=1}^{M} \sum_{m_2=m_1}^{M} .. \sum_{m_n=m_{n-1}}^{M} \frac{1}{(2i)^n \; n!} \sum_{\pi\in S_n} \times \\
&\times[\pi_\omega(W_\pm^{k_{\pi(1)}}(I_{m_{\pi(1)}})) - \pi_\omega(W_\pm^{-k_{\pi(1)}}(I_{m_{\pi(1)}}))] \, ..[\pi_\omega(W_\pm^{k_{\pi(n)}}(I_{m_{\pi(n)}})) - \pi(W_\pm^{-k_{\pi(n)}}(I_{m_{\pi(n)}}))]
\end{aligned} \tag{6.43}$$

where $\mathcal{P} = \{I_m;\; m = 1, .., M := |\mathcal{P}|\}$ is any partition of $[0, 2\pi]$ into consecutive intervals, e.g. $I_m = [(m-1)2\pi/n, m\pi/n]$ in some coordinate system, and we have again introduced a symmetric ordering in order that the corresponding operator be at least symmetric. Notice that the expression for $R_{\pm,\mathcal{P}}(0)$ had to arbitrarily choose a "starting interval" I_1 of $\mathcal{P}$ but the cyclic symmetrization in $Z_{\pm,\mathcal{P}}$ removes this arbitrariness again in analogy to the continuum expressions of section 3.

Expression (6.43) is an element of $\mathfrak{A}$ at finite M. It has the expected transformation behaviour under Lorentz transformations, namely $\alpha_L(Z_{\pm,\mathcal{P}}^{k_1..k_n}) = Z_{\pm,\mathcal{P}}^{L\cdot k_1..L\cdot k_n}$ and under Diff(S^1) the partition is changed to

its diffeomorphic image, i.e $\alpha^\pm_\varphi(Z^{k_1..k_n}_{\pm,\mathcal{P}}) = Z^{L\cdot k_1..L\cdot k_n}_{\pm,\varphi(\mathcal{P})}$, leaving M unchanged. The invariance property of the charges is hence only to be recovered in the limit $M \to \infty$, that is, $\mathcal{P} \to \sigma$. Hence one would like to define

$$\pi_\omega(Z^{k_1..k_n}_\pm) := \lim_{|\mathcal{P}| \to S^1} \pi_\omega(Z^{k_1..k_n}_{\pm,\mathcal{P}}) \tag{6.44}$$

in a suitable operator topology.

It is easy to see that with this definition we have $\pi_\omega(Z^{k_1..k_n}_\pm) = 0$ in the weak operator topology on $\mathcal{H}_\omega$ while $\pi_\omega(Z^{k_1..k_n}_\pm) = \infty$ in the strong operator topology on $\mathcal{H}_\omega$, the divergence being of the order of $M^{n/2}$. One can prove that this happens in general for diffeomorphism invariant operators in LQG [2, 26] which are "graph – changing", as the $\pi_\omega(Z_{\pm,\mathcal{P}})$ do when defined as above, if one wants to define them on the kinematical Hilbert space, in this case $\mathcal{H}_\omega$. This point will be explained in more detail in our companion paper [13]. Suffice it to say, for the purposes of the present paper, that there are two ways to get around this problem. The first one is to define the operators $Z_\pm$ on the physical Hilbert space (to be defined below), hence one uses the strong topology of the physical Hilbert space rather than the kinematical one which removes the above divergence which is due to the "infinite volume of the diffeomporphism group". We will follow this approach in the companion paper [13]. The second possibility is to define the operators $\pi_\omega(Z_\pm)$ directly on the kinematical Hilbert space $\mathcal{H}_\omega$ such that they are diffeomorphism invariant but simultaneously *non – graph changing*. This is the possibility which we will explore below.

The idea for defining a non – graph changing operator comes from looking at expression (6.43):
We would like to define $\pi_\omega(Z_\pm)$ densely on $\mathcal{H}_\omega$, hence it is sufficient to define it on the orthonormal basis given by the $\pi_\omega(W_\pm(s))\Omega_\omega$. Now given s we may assign to it a "distributional" smearing function

$$f^s_\mu(x) = \sum_{I \in \gamma(s)} \chi_I(x)\; k^I_\mu(s) \tag{6.45}$$

such that $\exp(iY_\pm(f^s)) = W_\pm(s)$. The point is now that if we allow some of the momenta $k^I(s)$ to be zero then we can consider the graph $\gamma(s)$ as a *partition* of S^1. The only condition on the momenta $k^I(s)$ is that momenta assigned to neighbouring intervals I are different from each other.

Now in the regularization $\pi_\omega(Z_{\pm,\mathcal{P}})$ of $\pi_\omega(Z_\pm)$ above we had to invoke a partition $\mathcal{P}$ as well, but any finite partition does not render $Z_{\pm,\mathcal{P}}$ into a diffeomorphism invariant object, it is only diffeomorphism covariant. We now we bring these two things together:
We simply define for $n \geq 2$

$$\pi_\omega(Z^{k_1..k_n}_\pm)\pi_\omega(W_\pm(s))\Omega_\omega := \pi_\omega(Z^{k_1..k_n}_{\pm,\gamma(s)})\pi_\omega(W_\pm(s))\Omega_\omega \tag{6.46}$$

with $\pi_\omega(Z^{(n)}_{\pm,0}) := 0$, the vacuum is annihilated by all charges. Also, if the decomposition of (6.46) into states of the form $\pi_\omega(W_\pm(s'))\Omega_\omega$ contains states with $\gamma(s')$ strictly smaller than $\gamma(s)$ then we simply remove that state from the decomposition which ensures that (6.46) becomes a symmetric operator as we will show in the next section.

Hence we have simply set $\mathcal{P} := \gamma(s)$ in (6.43). There is no limit $\mathcal{P} \to \infty$ to be taken, the partition is kept finite. The idea is that the semiclassical limit of (6.46) is reached only on states which have $|\gamma(s)|$ significantly large. This we will confirm below. We identify $\pi_\omega(Z^\mu_\pm)$ with the self – adjoint generator of the translation subgroup of the Poincaré group which is represented weakly continuously on $\mathcal{H}_\omega$. In particular, it commutes with all charges $\pi_\omega(Z^{(n)}_\pm)$ since it commutes with all the $\pi_\omega(W_\pm(s))$.

Let us check that (6.46) defines a diffeomorphism invariant operator. We must verify that

$$\begin{aligned} & U^\pm_\omega(\varphi)\pi_\omega(Z^{k_1..k_n}_\pm)\pi_\omega(W_\pm(s))\Omega_\omega = U^\pm_\omega(\varphi)\pi_\omega(Z^{k_1..k_n}_{\pm,\gamma(s)})\pi_\omega(W_\pm(s))\Omega_\omega \\ \equiv\; & \pi_\omega(Z^{k_1..k_n}_\pm)U^\pm_\omega(\varphi)\pi_\omega(W_\pm(s))\Omega_\omega = \pi_\omega(Z^{k_1..k_n}_{\pm,\varphi(\gamma(s))})\pi_\omega(\alpha^\pm_\varphi(W_\pm(s)))\Omega_\omega \end{aligned} \tag{6.47}$$

Now by construction the right hand side in the first line of (6.47) is a finite linear combination of vectors of the form $\pi_\omega(W_\pm(s'))\Omega_\omega$ with $\gamma(s') \subset \gamma(s)$ to which the unitary transformation $U^\pm_\omega(\varphi)$ is applied and hence these vectors are transformed into $\pi_\omega(W_\pm(\tilde{s}))\Omega_\omega$ with $\gamma(\tilde{s}) \subset \varphi(\gamma(s))$. But this reproduces exactly the action of the operator in the second line of (6.47) on the transformed states.

6.6 Properties of the Quantum Pohlmeyer Charges

In this section we wish to check that the algebra implied by (6.46) defines a quantum deformation of the classical invariant algebra of section 3.

Adjointness Relations:

Notice that the Hilbert space $\mathcal{H}_\omega^\pm$ can be written as an uncountable direct sum

$$\mathcal{H}_\omega^\pm = \overline{\oplus_\gamma \ \mathcal{H}_{\omega,\gamma}^\pm} \tag{6.48}$$

where the overline denotes completion, the sum is over all partitions (graphs) of S^1 and $\mathcal{H}_{\omega,\gamma}^\pm$ is the completion of the finite linear span of the vectors $\pi_\omega(W_\pm(s))$ with $\gamma(s)=\gamma$ with s such that for neighbouring I,J we have $k^I(s)\neq k^J(s)$.

Let $P_\gamma^\pm$ be the orthogonal projection onto $\mathcal{H}_{\omega\gamma}^\pm$. Then it is easy to see that

$$\pi_\omega(Z_\pm^{k_1..k_n}) = \oplus_\gamma \ P_\gamma^\pm \ \ \pi_\omega(Z_{\pm,\gamma}^{k_1..k_n}) \ \ P_\gamma^\pm \tag{6.49}$$

where the expression for $\pi_\omega(Z_{\pm,\gamma}^{k_1..k_n})$ is given in (6.46). It is now manifest that (6.49) defines a symmetric operator because $\pi_\omega(Z_{\pm,\gamma}^{k_1..k_n})$ is symmetric on $\mathcal{H}_{\omega,\gamma}^\pm$ provided it preserves γ which, however, is ensured by the projections. Since the expression (6.46) is real valued (it maps basis elements into finite linear combinations of basis elements with real valued components) it follows from von Neumann's involution theorem [35] that it has self-adjoint extensions. We do not need to worry about these extensions for what follows.

In order to define the $\pi_\omega(Z_\pm^{\mu_1..\mu_n})$ themselves we use the trivial observation that classically $Z_\pm^{k_1..k_n}L^n = Z_\pm^{\mu_1..\mu_n}$ if we set $k_{j\mu}=\delta_\mu^{\mu_j}/L$ where L is an arbitrary but fixed parameter of dimension cm^1. Hence we define

$$\pi_\omega(Z_\pm^{\mu_1..\mu_n}) := L^n \pi_\omega(Z_\pm^{k_1..k_n}), \ \ k_{j\mu} := \delta_\mu^{\mu_j}/L \tag{6.50}$$

The parameter L will enter the semiclassical analysis in section 6.7. We will see that (6.50) approximates the classical expression the better the smaller the parameter $t := (\ell_s/L)^2$ is.

Algebraic Properties:

We can now study the algebra of our $\pi_\omega(Z_\pm)$ and check that up to quantum corrections the classical algebra of invariants is reproduced on each of the invariant $\mathcal{H}_{\omega,\gamma}^\pm$ separately. To simplify the notation, let us introduce the shorthand

$$\pi_\omega(Y_\pm^k(I)) := \frac{1}{2i}[\pi_\omega(W_\pm^k(I)) - \pi_\omega(W_\pm^{-k}(I))] \tag{6.51}$$

corresponding to (6.42). In what follows we consider the algebra of the $\pi_\omega(Z_\pm)$ restricted to a fixed $\mathcal{H}_{\omega,\gamma}^\pm$ with $|\gamma|=M$ and $\gamma=\{I_m\}_{m=1}^M$ where we have chosen an arbitrary starting interval I_1 and the intervals $I_m, \ I_{m+1}$ with $m\equiv m+M$ are next neighbours. We will denote the corresponding restrictions by

$$\begin{aligned}
\pi_\omega(Z_{\pm,M}^{k_1..k_n}) &= \pi_\omega(R_{\pm,M}^{k_1..k_n}) + \pi_\omega(R_{\pm,M}^{k_2..k_nk_1}) + \ .. \ + \pi_\omega(R_{\pm,M}^{k_nk_1..k_{n-1}}) \\
\pi_\omega(R_{\pm,M}^{k_1..k_n}) &= \sum_{1\leq m_1\leq ..\leq m_n\leq M} \frac{1}{n!}\sum_{\pi\in S_n} \pi_\omega(Y_\pm^{k_{\pi(1)}}(I_{m_{\pi(1)}})) \ .. \ \pi_\omega(Y_\pm^{k_{\pi(n)}}(I_{m_{\pi(n)}}))
\end{aligned} \tag{6.52}$$

We now notice that (6.52) is the quantization of a Riemann sum approximation for the classical continuum integrals of section 3 (of course, the approximation is classically good only for large M). Since algebraic properties of iterated path ordered integrals have a precise analog for iterated path ordered Riemann sums, it is clear that all algebraic relations of section 3 which only rely on manipulations of integrals are exactly mirrored by expressions (6.52) up to quantum corrections which come from 1. commuting products of the

$\pi_\omega(Y^k_\pm(I))$, 2. ommission of states due to the projections in (6.49) and 3. "finite size effects" which come from the fact that we are dealing really with discrete objects (sums and intervals) rather than continuum ones (integrals and points). All of these corrections are suppressed in the semiclassical limit which is reached for states with large M (they are, in a precise sense, of measure zero) and in the limit that $\ell_s \to 0$ as we will demonstrate below. Instead of going through a tedious bookkeeping exercise which would merely reproduce the results of [7, 8, 9, 20] in a discrete language while keeping track of the operator ordering, let us give a typical example which illustrates these effects.

Notice that

$$\begin{aligned}
&[\pi_\omega(Y^k_\pm(I_m)),\pi_\omega(Y^l_\pm(I_{m'}))] = \frac{-i}{2}\sin(\mp\ell_s^2[k\cdot l]\alpha(I_m,I_{m'})/2)\times \\
&\times\{\pi_\omega(W_\pm((k,I_m)+(l,I_{m'}))+\pi_\omega([W_\pm((k,I_m)+(l,I_{m'})]^{-1}) \\
&+\{\pi_\omega(W_\pm((k,I_m)+(-l,I_{m'}))+\pi_\omega([W_\pm((k,I_m)+(-l,I_{m'})]^{-1})\}
\end{aligned} \tag{6.53}$$

where

$$\alpha(I_m,I_{m'}) = (\chi_{I_m})_{|\partial I_{m'}} - (\chi_{I_{m'}})_{|\partial I_m} = \begin{cases} 0 & m=m' \text{ or } |m-m'|>1 \\ -1 & m=m'+1 \\ 1 & m'=m+1 \end{cases} \tag{6.54}$$

where we have used the convention that $\chi_I(x)=1$ for $x\in I-\partial I$, $\chi_I(x)=1/2$ for $x\in\partial I$ and $\chi_I(x)=0$ for $I\notin I$. This convention coincides $dx-$ a.e. with the usual convention but in our case does make a difference due to the singular support of our fields. It is the unique convention which ensures that for closed intervals I,J with $f(I)=b(J)$ we have $\chi_{I\cup J}=\chi_I+\chi_J$. Here $b(I),f(I)$ denote beginning point and final point of I respectively.

Hence

$$\begin{aligned}
&[\pi_\omega(Y^k_\pm(I_m)),\pi_\omega(Y^l_\pm(I_{m'}))] = \mp\frac{i}{2}\sin(\ell_s^2[k\cdot l]/2)(\delta_{m,m'+1}-\delta_{m,m'-1})\times \\
&\times\{\pi_\omega(W_\pm((k,I_m)+(l,I_{m'}))+\pi_\omega([W_\pm((k,I_m)+(l,I_{m'})]^{-1}) \\
&+\pi_\omega(W_\pm((k,I_m)+(-l,I_{m'}))+\pi_\omega([W_\pm((k,I_m)+(-l,I_{m'})]^{-1})\} \\
=:\quad &\mp 2i\sin(\ell_s^2[k\cdot l]/2)[\delta_{m,m'+1}-\delta_{m,m'-1}]\pi_\omega(a((m,k),(m',l))
\end{aligned} \tag{6.55}$$

Semiclassicaly the expression $\pi_\omega(a((m,k),(m',l))$ will tend to the constant 1 so that (6.55) is a specific quantum deformation of the classical Poisson bracket.

As an example we choose (notice that the first charge algebraically indpendent of p_μ involves three indices because $Z^{\mu_1\mu_2}=Z^{\mu_1}Z^{\mu_2}/2$ however, in the following illustrational calculation we will not make use of this fact)

$$
\begin{aligned}
\pi_\omega(Z^{k_1}_{\pm,M})\pi_\omega(Z^{k_2k_3}_{\pm,M}) \;=\;& \frac{1}{2}\sum_{m_1=1}^{M}\sum_{1\le m_2\le m_3\le M}\pi_\omega(Y^{k_1}_\pm(I_{m_1}))\times\\
&\times[\pi_\omega(Y^{k_2}_\pm(I_{m_2}))\pi_\omega(Y^{k_3}_\pm(I_{m_3}))+\pi_\omega(Y^{k_3}_\pm(I_{m_3}))\pi_\omega(Y^{k_2}_\pm(I_{m_2}))\\
&+\pi_\omega(Y^{k_3}_\pm(I_{m_2}))\pi_\omega(Y^{k_2}_\pm(I_{m_3}))+\pi_\omega(Y^{k_2}_\pm(I_{m_3}))\pi_\omega(Y^{k_3}_\pm(I_{m_2}))]\\
=\;& \frac{1}{2}\sum_{1\le m_1\le m_2\le m_3\le M}\pi_\omega(Y^{k_1}_\pm(I_{m_1}))\times\\
&\times[\pi_\omega(Y^{k_2}_\pm(I_{m_2}))\pi_\omega(Y^{k_3}_\pm(I_{m_3}))+\pi_\omega(Y^{k_3}_\pm(I_{m_3}))\pi_\omega(Y^{k_2}_\pm(I_{m_2}))\\
&+\pi_\omega(Y^{k_3}_\pm(I_{m_2}))\pi_\omega(Y^{k_2}_\pm(I_{m_3}))+\pi_\omega(Y^{k_2}_\pm(I_{m_3}))\pi_\omega(Y^{k_3}_\pm(I_{m_2}))]\\
+\;& \frac{1}{2}\sum_{1\le m_2< m_1\le m_3\le M}\pi_\omega(Y^{k_1}_\pm(I_{m_1}))\times\\
&\times[\pi_\omega(Y^{k_2}_\pm(I_{m_2}))\pi_\omega(Y^{k_3}_\pm(I_{m_3}))+\pi_\omega(Y^{k_3}_\pm(I_{m_3}))\pi_\omega(Y^{k_2}_\pm(I_{m_2}))\\
&+\pi_\omega(Y^{k_3}_\pm(I_{m_2}))\pi_\omega(Y^{k_2}_\pm(I_{m_3}))+\pi_\omega(Y^{k_2}_\pm(I_{m_3}))\pi_\omega(Y^{k_3}_\pm(I_{m_2}))]\\
+\;& \frac{1}{2}\sum_{1\le m_2\le m_3< m_1\le M}\pi_\omega(Y^{k_1}_\pm(I_{m_1}))\times\\
&\times[\pi_\omega(Y^{k_2}_\pm(I_{m_2}))\pi_\omega(Y^{k_3}_\pm(I_{m_3}))+\pi_\omega(Y^{k_3}_\pm(I_{m_3}))\pi_\omega(Y^{k_2}_\pm(I_{m_2}))\\
&+\pi_\omega(Y^{k_3}_\pm(I_{m_2}))\pi_\omega(Y^{k_2}_\pm(I_{m_3}))+\pi_\omega(Y^{k_2}_\pm(I_{m_3}))\pi_\omega(Y^{k_3}_\pm(I_{m_2}))]\\
=\;& \frac{1}{3!}\sum_{1\le m_1\le m_2\le m_3\le M}[3\pi_\omega(Y^{k_1}_\pm(I_{m_1}))]\times\\
&\times[\pi_\omega(Y^{k_2}_\pm(I_{m_2}))\pi_\omega(Y^{k_3}_\pm(I_{m_3}))+\pi_\omega(Y^{k_3}_\pm(I_{m_3}))\pi_\omega(Y^{k_2}_\pm(I_{m_2}))\\
&+\pi_\omega(Y^{k_3}_\pm(I_{m_2}))\pi_\omega(Y^{k_2}_\pm(I_{m_3}))+\pi_\omega(Y^{k_2}_\pm(I_{m_3}))\pi_\omega(Y^{k_3}_\pm(I_{m_2}))]\\
+\;& \frac{1}{3!}\sum_{1\le m_2\le m_1\le m_3\le M}[3\pi_\omega(Y^{k_1}_\pm(I_{m_1}))]\times\\
&\times[\pi_\omega(Y^{k_2}_\pm(I_{m_2}))\pi_\omega(Y^{k_3}_\pm(I_{m_3}))+\pi_\omega(Y^{k_3}_\pm(I_{m_3}))\pi_\omega(Y^{k_2}_\pm(I_{m_2}))\\
&+\pi_\omega(Y^{k_3}_\pm(I_{m_2}))\pi_\omega(Y^{k_2}_\pm(I_{m_3}))+\pi_\omega(Y^{k_2}_\pm(I_{m_3}))\pi_\omega(Y^{k_3}_\pm(I_{m_2}))]\\
+\;& \frac{1}{3!}\sum_{1\le m_2\le m_3\le m_1\le M}[3\pi_\omega(Y^{k_1}_\pm(I_{m_1}))]\times\\
&\times[\pi_\omega(Y^{k_2}_\pm(I_{m_2}))\pi_\omega(Y^{k_3}_\pm(I_{m_3}))+\pi_\omega(Y^{k_3}_\pm(I_{m_3}))\pi_\omega(Y^{k_2}_\pm(I_{m_2}))\\
&+\pi_\omega(Y^{k_3}_\pm(I_{m_2}))\pi_\omega(Y^{k_2}_\pm(I_{m_3}))+\pi_\omega(Y^{k_2}_\pm(I_{m_3}))\pi_\omega(Y^{k_3}_\pm(I_{m_2}))]\\
-\;& \frac{1}{2}\sum_{1\le m_2\le m_3\le M}\pi_\omega(Y^{k_1}_\pm(I_{m_2}))\times\\
&\times[\pi_\omega(Y^{k_2}_\pm(I_{m_2}))\pi_\omega(Y^{k_3}_\pm(I_{m_3}))+\pi_\omega(Y^{k_3}_\pm(I_{m_3}))\pi_\omega(Y^{k_2}_\pm(I_{m_2}))\\
&+\pi_\omega(Y^{k_3}_\pm(I_{m_2}))\pi_\omega(Y^{k_2}_\pm(I_{m_3}))+\pi_\omega(Y^{k_2}_\pm(I_{m_3}))\pi_\omega(Y^{k_3}_\pm(I_{m_2}))]\\
-\;& \frac{1}{2}\sum_{1\le m_2\le m_3\le M}\pi_\omega(Y^{k_1}_\pm(I_{m_3}))\times\\
&\times[\pi_\omega(Y^{k_2}_\pm(I_{m_2}))\pi_\omega(Y^{k_3}_\pm(I_{m_3}))+\pi_\omega(Y^{k_3}_\pm(I_{m_3}))\pi_\omega(Y^{k_2}_\pm(I_{m_2}))\\
&+\pi_\omega(Y^{k_3}_\pm(I_{m_2}))\pi_\omega(Y^{k_2}_\pm(I_{m_3}))+\pi_\omega(Y^{k_2}_\pm(I_{m_3}))\pi_\omega(Y^{k_3}_\pm(I_{m_2}))]
\end{aligned}
\tag{6.56}
$$

The first three terms in the last equality of (6.55) combine, up to commutators, to the expected expression $\pi_\omega(Z^{k_1k_2k_3}_\pm)+\pi_\omega(Z^{k_2k_1k_3}_\pm)$, see section 3, while the two remaining terms converge in the semiclassical limit

to path ordered integrals of the form

$$\frac{1}{M}\int_{x_1\le x_2} d^2x Y_\pm^{k_1}(x_2)Y_\pm^{k_2}(x_2)Y_\pm^{k_3}(x_3) \tag{6.57}$$

and thus vanishes in the large M limit, see below. For the general relations we get similar correction terms whose number depends only on n and which are therefore suppressed compared to the correct leading term as $M\to\infty$ and $\ell_s\to 0$.

Commutation Relations:

Next we consider commutators. We have

$$\begin{aligned}
&[\pi_\omega(Z_\pm^{k_1..k_n}),\pi_\omega(Z_\pm^{k'_1..k'_{n'}})] = C_n\cdot S_n\cdot \sum_{1\le m_1\le..\le m_n\le M} C_{n'}\cdot S_{n'}\cdot \sum_{1\le m'_1\le..\le m'_{n'}\le M} \times \\
&\times[\pi_\omega(Y_\pm^{k_1}(I_{m_1})\;..\;\pi_\omega(Y_\pm^{k_n}(I_{m_n})),\pi_\omega(Y_\pm^{k'_1}(I_{m'_1}))\;..\;\pi_\omega(Y_\pm^{k'_{n'}}(I_{m'_{n'}}))] \\
= &\mp 2i\sum_{l=1}^{n}\sum_{l'=1}^{n'}\sin(\ell_s^2[k_l\cdot k'_{l'}]/2)\times \\
&\times C_n\cdot S_n\cdot\sum_{m_1,..,m_n=1}^{M}\theta(m_2-m_1)..\theta(m_n-m_{n-1})\;C_{n'}\cdot S_{n'}\cdot\sum_{m'_1,..,m'_{n'}=1}^{M}\theta(m'_2-m'_1)..\theta(m'_{n'}-m'_{n'-1})\;\times \\
&\times[\delta_{m_l,m'_{l'}+1}-\delta_{m'_{l'},m_l+1}]\times \\
&\times\pi_\omega(Y_\pm^{k_1}(I_{m_1}))\;..\;\pi_\omega(Y_\pm^{k_{l-1}}(I_{m_{l-1}}))\pi_\omega(Y_\pm^{k'_1}(I_{m'_1}))\;..\;\pi_\omega(Y_\pm^{k'_{l'-1}}(I_{m'_{l'-1}}))\pi_\omega(a((I_{m_l},k_{m_l}),(I_{m'_{l'}},k'_{m'_{l'}})))\times \\
&\times\pi_\omega(Y_\pm^{k'_{l'+1}}(I_{m'_{l'+1}}))\;..\;\pi_\omega(Y_\pm^{k'_{n'}}(I_{m'_{n'}}))\pi_\omega(Y_\pm^{k_{l+1}}(I_{m_{l+1}}))\;..\;\pi_\omega(Y_\pm^{k_n}(I_{m_n})) \\
= &\mp 2i\sum_{l=1}^{n}\sum_{l'=1}^{n'}\sin(\ell_s^2[k_l\cdot k'_{l'}]/2)\times \\
&\times C_n\cdot S_n\cdot\sum_{m_1,..,\hat{m}_l,..,m_n=1}^{M}\theta(m_2-m_1)\;..\;\theta(m_{l-1}-m_{l-2})\;\theta(m_{l+2}-m_{l+1})\;..\theta(m_n-m_{n-1})\;\times \\
&\times C_{n'}\cdot S_{n'}\cdot\sum_{m'_1,..,\hat{m}'_{l'},..,m'_{n'}=1}^{M}\theta(m'_2-m'_1)\;..\;\theta(m'_{l'-1}-m'_{l'-2})\;\theta(m'_{l'+2}-m'_{l'+1})\;..\theta(m'_{n'}-m'_{n'-1})\;\times \\
&\times\pi_\omega(Y_\pm^{k_1}(I_{m_1}))\;..\;\pi_\omega(Y_\pm^{k_{l-1}}(I_{m_{l-1}}))\pi_\omega(Y_\pm^{k'_1}(I_{m'_1}))\;..\;\pi_\omega(Y_\pm^{k'_{l'-1}}(I_{m'_{l'-1}}))\times \\
&\times\{\sum_{m_l,m'_{l'}=1}^{M}[(\delta_{m_l,m'_{l'}+1}-\delta_{m_l,m'_{l'}})-(\delta_{m'_{l'},m_l+1}-\delta_{m_l,m'_{l'}})]\;\times \\
&\times\theta(m_l-m_{l-1})\theta(m_{l+1}-m_l)\theta(m'_{l'}-m'_{l'-1})\theta(m'_{l'+1}-m'_{l'})\;\pi_\omega(a((I_{m_l},k_{m_l}),(I_{m'_{l'}},k'_{m'_{l'}}))\}\times \\
&\times\pi_\omega(Y_\pm^{k'_{l'+1}}(I_{m'_{l'+1}}))\;..\;\pi_\omega(Y_\pm^{k'_{n'}}(I_{m'_{n'}}))\pi_\omega(Y_\pm^{k_{l+1}}(I_{m_{l+1}}))\;..\;\pi_\omega(Y_\pm^{k_n}(I_{m_n}))
\end{aligned} \tag{6.58}$$

where C_n and S_n respectively enforce cyclic summation of the $k_1..k_n$ and symmetric projection of the $((m_1,k_1),..,(m_n,k_n))$ respectively. Here θ is the Heavyside step function and a hat above the argument denotes its omission.

Consider the curly bracket in the last equality of (6.58). We see that the square bracket that it contains is precisely the discretization of the distribution $(\partial_y-\partial_x)\delta(x,y)$ of the corresponding continuum calculation

where the derivatives and $\delta-$distributions respectively are replaced by differences and Kronecker δ functions respectively. In the continuum calculation we would now perform an integration by parts, in the discrete calculation we perform a partial resummation. We have

$$
\begin{aligned}
& \sum_{m,m'=1}^{M} [(\delta_{m,m'+1}-\delta_{m,m'})-(\delta_{m+1,m'}-\delta_{m,m'})]f(m,m') \\
= & \sum_{m=1}^{M}[-f(m,M)\delta_{m,M}+\sum_{m'=1}^{M-1}(\delta_{m,m'+1}-\delta_{m,m'})f(m,m') \\
& -\sum_{m'=1}^{M}[-f(M,m')\delta_{M,m'}+\sum_{m=1}^{M-1}(\delta_{m+1,m'}-\delta_{m,m'})f(m,m') \\
= & \sum_{m=1}^{M}[\sum_{m'=2}^{M}\delta_{m,m'}f(m,m'-1)-\sum_{m'=1}^{M-1}\delta_{m,m'}f(m,m')] \\
& -\sum_{m'=1}^{M}[\sum_{m=2}^{M}\delta_{m,m'}f(m-1,m')\sum_{m=1}^{M-1}\delta_{m,m'})f(m,m')] \\
= & \sum_{m=1}^{M}[\delta_{m,M}f(m,M)+\sum_{m'=2}^{M}\delta_{m,m'}(f(m,m'-1)-f(m,m'))] \\
& -\sum_{m'=1}^{M}[\delta_{M,m'}f(M,m')+\sum_{m=2}^{M}\delta_{m,m'}(f(m-1,m')-f(m,m'))] \\
= & \sum_{m,m'=1}^{M}\delta_{m,m'}\{[f(m,m'-1)-f(m,m')]-[f(m-1,m')-f(m,m')]\}
\end{aligned}
\tag{6.59}
$$

for any function defined on $\{1,..,M\}^2$ where we used the convention $f(m,m')=0$ if one of m,m' equals $0,M+1$.

Now for products of (possibly non commutative) functions defined on discrete values the difference replacing the derivative gives a discrete version of the Leibniz rule

$$[fg](m-1)-[fg](m)=[f(m-1)-f(m)]g(m-1)+f(m)[g(m-1)-g(m)] \tag{6.60}$$

Combining (6.59) and (6.60) we may write the curly bracket in the last equality of (6.58) as

$$
\begin{aligned}
& \sum_{m_l,m'_{l'}=1}^{M} [(\delta m_l, m'_{l'}+1-\delta m_l, m'_{l'}) - (\delta m'_{l'}, m_l+1-\delta m_l, m'_{l'}] \ \times \qquad (6.61) \\
& \times\theta(m_l-m_{l-1})\theta(m_{l+1}-m_l)\theta(m'_{l'}-m'_{l'-1})\theta(m'_{l'+1}-m'_{l'}) \ \pi_\omega(a((I_{m_l},k_{m_l}),(I_{m'_{l'}},k'_{m'_{l'}})) \\
= & \sum_{m_l,m'_{l'}=1}^{M} \delta_{m_l,m'_{l'}} \ \{\theta(m_l-m_{l-1})\theta(m_{l+1}-m_l) \ \times \\
& \times\{[\theta(.-m'_{l'-1})\theta(m'_{l'+1}-.)\pi_\omega(a((I_{m_l},k_{m_l}),(I_.,k'_.))](m'_{l'}-1) \\
& -[\theta(.-m'_{l'-1})\theta(m'_{l'+1}-.)\pi_\omega(a((I_{m_l},k_{m_l}),(I_.,k'_.))](m'_{l'})\} \\
& -\theta(m'_{l'}-m'_{l'-1})\theta(m'_{l'+1}-m'_{l'}) \ \times \\
& \times\{[\theta(.-m_{l-1})\theta(m_{l+1}-.)\pi_\omega(a((I_.,k_.),(I_{m'_{l'}},k'_{m'_{l'}}))](m_l-1) \\
& -[\theta(.-m_{l-1})\theta(m_{l+1}-.)\pi_\omega(a((I_.,k_.),(I_{m'_{l'}},k'_{m'_{l'}}))](m_l)\}\} \\
= & \sum_{m_l,m'_{l'}=1}^{M} \delta_{m_l,m'_{l'}} \ \{\theta(m_l-m_{l-1})\theta(m_{l+1}-m_l) \ \times \\
& \times\{[(\theta(.-m'_{l'-1})\theta(m'_{l'+1}-.))(m'_{l'}-1)-(\theta(.-m'_{l'-1})\theta(m'_{l'+1}-.))(m'_{l'})]\pi_\omega(a((I_{m_l},k_{m_l}),(I_.,k'_.))](m'_{l'}-1) \\
& +(\theta(.-m'_{l'-1})\theta(m'_{l'+1}-.))(m'_{l'})[\pi_\omega(a((I_{m_l},k_{m_l}),(I_.,k'_.))(m'_{l'}-1)-\pi_\omega(a((I_{m_l},k_{m_l}),(I_.,k'_.))(m'_{l'})]\} \\
& -\theta(m'_{l'}-m'_{l'-1})\theta(m'_{l'+1}-m'_{l'}) \ \times \\
& \times\{[(\theta(.-m_{l-1})\theta(m_{l+1}-.))(m_l-1)-(\theta(.-m_{l-1})\theta(m_{l+1}-.))(m_l)]\pi_\omega(a((I_.,k_.),(I_{m'_{l'}},k'_{m'_{l'}}))](m_l-1) \\
& +(\theta(.-m_{l-1})\theta(m_{l+1}-.))(m_l)[\pi_\omega(a((I_.,k_.),(I_{m'_{l'}},k'_{m'_{l'}}))(m_l-1)-\pi_\omega(a((I_.,k_.),(I_{m'_{l'}},k'_{m'_{l'}}))(m_l)]\}\}
\end{aligned}
$$

Now

$$
\begin{aligned}
& (\theta(.-m_{l-1})\theta(m_{l+1}-.))(m_l-1)-(\theta(.-m_{l-1})\theta(m_{l+1}-.))(m_l) \\
= & [\theta(m_l-1-m_{l-1})-\theta(m_l-m_{l-1})]\theta(m_{l+1}-(m_l-1))+\theta(m_l-m_{l-1})[\theta(m_{l+1}-(m_l-1))-\theta(m_{l+1}-m_l)] \\
= & -\delta_{m_l,m_{l-1}}\theta(m_{l+1}-(m_l-1))+\delta_{m_l,m_{l+1}+1}\theta(m_l-m_{l-1}) \qquad (6.62)
\end{aligned}
$$

and similarly for the primed quantities. This allows us to carry out the sum over $m_l, m'_{l'}$ in the first and third term of (6.61) which simplifies to

$$
\begin{aligned}
& \sum_{m_l,m'_{l'}=1}^{M} \delta_{m_l,m'_{l'}} \ \{\theta(m_l-m_{l-1})\theta(m_{l+1}-m_l) \ \times \qquad (6.63) \\
& \times\{[-\delta_{m'_{l'},m'_{l'-1}}\theta(m'_{l'+1}-(m'_{l'}-1))+\delta_{m'_{l'},m'_{l'+1}+1}\theta(m'_{l'}-m'_{l'-1})]\pi_\omega(a((I_{m_l},k_{m_l}),(I_.,k'_.))](m'_{l'}-1) \\
& +(\theta(.-m'_{l'-1})\theta(m'_{l'+1}-.))(m'_{l'})[\pi_\omega(a((I_{m_l},k_{m_l}),(I_.,k'_.))(m'_{l'}-1)-\pi_\omega(a((I_{m_l},k_{m_l}),(I_.,k'_.))(m'_{l'})]\} \\
& -\theta(m'_{l'}-m'_{l'-1})\theta(m'_{l'+1}-m'_{l'}) \ \times \\
& \times\{[-\delta_{m_l,m_{l-1}}\theta(m_{l+1}-(m_l-1))+\delta_{m_l,m_{l+1}+1}\theta(m_l-m_{l-1})]\pi_\omega(a((I_.,k_.),(I_{m'_{l'}},k'_{m'_{l'}}))](m_l-1) \\
& +(\theta(.-m_{l-1})\theta(m_{l+1}-.))(m_l)[\pi_\omega(a((I_.,k_.),(I_{m'_{l'}},k'_{m'_{l'}}))(m_l-1)-\pi_\omega(a((I_.,k_.),(I_{m'_{l'}},k'_{m'_{l'}}))(m_l)]\}\}
\end{aligned}
$$

Now when comparing with the continuum calculation, (6.63) is almost exactly the discrete counterpart of the result that one gets when integrating the derivatives of the $\delta-$distributions, coming from the Poisson brackets, by parts. The derivatives then hit the $\theta-$functions which results in a second $\delta-$distribution since $\theta'(x) = \delta(x)$. The only difference is that in the classical theory the operator $\pi_\omega(a(.,.,.,.))$ is replaced by the constant 1 so that the second and fourth term in (6.63) are missing. However, as we will see below, in a

semiclassical state the difference between (in the sense of expectation values) the operator and the constant is of order $1/M$ and hence is suppressed semiclassically. Thus, dropping the extra terms, up to quantum corrections which are the result from reordering terms, the above discrete calculation precisely reproduces the classical continuum calculation. In particular, after carrying out the sum over the Kronecker δ's we obtain from (6.63), dropping boundary terms of order $1/M$

$$\begin{aligned}
&\theta(m'_{l'+1}-m'_{l'-1}+1)[\theta(m'_{l'+1}-m_{l-1}+1)\theta(m_{l+1}-m'_{l'+1}-1)-\theta(m'_{l'-1}-m_{l-1})\theta(m_{l+1}-m'_{l'-1})] \\
&-\theta(m_{l+1}-m_{l-1}+1)[\theta(m_{l+1}-m'_{l'-1}+1)\theta(m'_{l'+1}-m_{l+1}-1)-\theta(m_{l-1}-m'_{l'-1})\theta(m'_{l'+1}-m_{l-1})]
\end{aligned} \tag{6.64}$$

The effect is thus that the path ordering of the primed and unprimed labels gets intermingled. One now inserts a unity

$$1=\sum_{\pi\in S_{n+n'-2}}\theta(p_{\pi(2)}-p_{pi(1)})..\theta(p_{\pi(n+n'-2)}-p_{\pi(n+n'-3)}) \tag{6.65}$$

where $(p_1,..,p_{n+n'-2})=(m_1,..,\hat{m}_l,..,m_n,m'_1,..,\hat{m}'_{l'},..,m'_{n'})$ and notices that due to the partly still existing projection on the orderings $m_1\le..\le m_n$ and $m'_1\le..\le m'_{n'}$ only the reshuffle sums displayed in section 3 survive. The remaining calculation is thus identical to the continuum for which we refer the reader to [7, 8, 9, 20].

6.7 Classical Limit of the Quantum Pohlmeyer Charges

Thus, we have verified that $\mathcal{H}_\omega$ carries a representation of the **Quantum Pohlmeyer Algebra** with precise quantum corrections provided we show that these corrections are subleading in the semiclassical limit of the theory. We will now show that this is actually the case. To do this we use the background independent semiclassical techniques developed in [36]:

We choose a graph γ with large $M=|\gamma|$ and use the same parameter L with the dimension of length that we used in (6.50). Then we consider the set $\mathcal{S}_{\gamma,L}$ of momentum network labels s such that $\gamma(s)=\gamma$ and such that $n^I_\mu(s):=k^I_\mu(s)L$ is an integer for every $\mu=0,..,D-1$ and every $I\in\gamma$. Given a point $m_0:=(\pi^0_\mu(x),X^\mu_0(x))_{x\in S^1}$ in the classical phase space $\mathcal{M}$ we construct the following quantities for $s\in\mathcal{S}_\gamma$

$$W_\pm(s,m_0):=\exp(i\sum_{I\in\gamma}k^I_\mu(s)\int_I dx\;[\pi^\mu_0(x)\pm X^{\mu\prime}_0(x)]) \tag{6.66}$$

We can rewrite (6.66) in the form

$$W_\pm(s,m_0)=\prod_{\mu=0}^{D-1}\prod_{I\in\gamma}[W_\pm(\mu,I,m_0)]^{n^I_\mu},\;\; W_\pm(\mu,I,m_0)=\exp(\int_I dx\;[\pi^\mu_0(x)+iX^{\mu\prime}(x)]/L) \tag{6.67}$$

where $k^I_\mu(s)=n^I_\mu/L$, $n^I_\mu\in\mathbb{Z}$. Next we choose any DM real numbers r^I_μ such that $r^I_\mu-r^J_\mu\notin\mathbb{Q}$ for $I\cap J\subset\partial I$ and $\mu=0,..,D-1$ and denote the corresponding momentum network labels with momenta $k^I_{\mu 0}:=r^L_\mu/L$ by s_0.

We now define a semiclassical state by

$$\psi^\pm_{\gamma,L,m_0}:=\sum_{s\in\mathcal{S}_{\gamma,L}}e^{-t\lambda(s)/2}\;\overline{W_\pm(s)}\;\pi_\omega(W_\pm(s+s_0))\Omega_\omega \tag{6.68}$$

where

$$t:=(\frac{\ell_s}{L})^2,\;\;\lambda(s)=\sum_{\mu=0}^{D-1}\sum_{I\in\gamma}(n^I_\mu)^2 \tag{6.69}$$

For the motivation to consider precisely those states see [36] or our companion paper [13]. The states (6.68) are normalizable due to the damping factor as we will see but not normalized. The parameter t is called the "classicality parameter" for reasons that will become obvious in a moment. Notice that our choices imply that every state in the infinite sum in (6.69) really has γ as the underlying graph.

When we apply $\pi_\omega(Z_\pm)$ to (6.69) we obtain a specific linear combination of operators of the form $\pi_\omega(W_\pm(s))$ with s in the "lattice" $\mathcal{S}_{\gamma,L}$. Again, due to the translation by s_0 in (6.68) none of these operators maps us out of $\mathcal{H}^\pm_{\omega,\gamma}$ so that the projections in (6.49) act trivially. Let us compute the action of these operators on our semiclassical states. We find

$$\begin{aligned}
\pi_\omega(W_\pm(s))\psi^\pm_{\gamma,L,m_0} &= \sum_{s'\in\mathcal{S}_{\gamma,L}} e^{-\frac{t}{2}\lambda(s')}\overline{W_\pm(s',m_0)}\; e^{\mp i\ell_s^2\alpha(s,s_0+s')/2}\;\pi_\omega(W_\pm(s+s'+s_0))\Omega_\omega \\
&= \sum_{s'\in\mathcal{S}_{\gamma,L}} e^{-\frac{t}{2}\lambda(s'-s)}\overline{W_\pm(s'-s,m_0)}\; e^{\mp i\ell_s^2\alpha(s,s_0+s'-s)/2}\;\pi_\omega(W_\pm(s'+s_0))\Omega_\omega \\
&= W_\pm(s,m_0)\; e^{\mp i\ell_s^2\alpha(s,s_0)/2}\sum_{s'\in\mathcal{S}_{\gamma,L}} e^{-\frac{t}{2}\lambda(s'-s)}\overline{W_\pm(s',m_0)}\; e^{\mp i\ell_s^2\alpha(s,s')/2}\;\pi_\omega(W_\pm(s'+s_0))\Omega_\omega
\end{aligned} \tag{6.70}$$

where we have used translation invariance of the lattice and antisymmetry as well as bilinearity of the function

$$\alpha(s,s') := \sum_{I\in\gamma(s),\; I'\in\gamma(s')} [k^I(s)\cdot k^{I'}(s')]\;\alpha(I,I') \tag{6.71}$$

We thus find for the expectation value

$$\begin{aligned}
&\frac{<\psi^\pm_{\gamma,L,m_0},\pi_\omega(W_\pm(s))\psi^\pm_{\gamma,L,m_0}>}{||\psi^\pm_{\gamma,L,m_0}||^2} \\
&= W_\pm(s,m_0)\; e^{\mp i\ell_s^2\alpha(s,s_0)/2}\;\frac{\sum_{s'\in\mathcal{S}_{\gamma,L}}\; e^{-\frac{t}{2}[\lambda(s'-s)+\lambda(s')]}\; e^{\mp i\ell_s^2\alpha(s,s')/2}}{\sum_{s'\in\mathcal{S}_{\gamma,L}}\; e^{-t\lambda(s')}}
\end{aligned} \tag{6.72}$$

In order to estimate this expression, let us write

$$\mp i\ell_s^2\alpha(s,s')/2 = \pm i\frac{t}{2}\sum_{I,\mu} n^I_\mu(s')[\sum_J n^{J\mu}(s)\alpha(I,J)] =: \pm i\frac{t}{2}\sum_{I,\mu} n^I_\mu(s')c^\mu_I(s) \tag{6.73}$$

so that (6.72) factorizes

$$\begin{aligned}
&\frac{<\psi^\pm_{\gamma,L,m_0},\pi_\omega(W_\pm(s))\psi^\pm_{\gamma,L,m_0}>}{||\psi^\pm_{\gamma,L,m_0}||^2} \\
&= W_\pm(s,m_0)\; e^{\mp i\ell_s^2\alpha(s,s_0)/2}\; e^{-\frac{t}{2}\lambda(s)}\;\prod_{\mu,I}\frac{\sum_{l\in\mathbb{Z}} e^{-t(l^2-l[n^I_\mu(s)\pm ic^\mu_I(s)/2])}}{\sum_{l\in\mathbb{Z}} e^{-tl^2}}
\end{aligned} \tag{6.74}$$

We are interested in the limit of small t and large M of this expression. In order to estimate it, the presentation (6.74) is not very useful because the series in both numerator and denominator converge only

slowly. Hence we apply the Poisson summation formula [36] and transform (6.74) into

$$
\begin{aligned}
&\frac{<\psi^{\pm}_{\gamma,L,m_0},\pi_\omega(W_\pm(s))\psi^{\pm}_{\gamma,L,m_0}>}{||\psi^{\pm}_{\gamma,L,m_0}||^2} && (6.75)\\
=\;& W_\pm(s,m_0)\,e^{\mp i\ell_s^2\alpha(s,s_0)/2}\,e^{-\frac{t}{2}\lambda(s)}\prod_{\mu,I} e^{\frac{(t[n^I_\mu(s)\pm ic^\mu_I(s)/2])^2}{4t}}\frac{\sum_{l\in\mathbb{Z}} e^{-\frac{\pi^2 l^2}{t}}\,e^{-2\pi l[n^I_\mu(s)\pm ic^\mu_I(s)/2]}}{\sum_{l\in\mathbb{Z}} e^{-\frac{\pi^2l^2}{t}}}\\
=\;& W_\pm(s,m_0)\,e^{\mp i\ell_s^2\alpha(s,s_0)/2}\,e^{-\frac{t}{2}\lambda(s)}\,e^{\frac{t}{4}\sum_{\mu,I}[n^I_\mu(s)\pm ic^\mu_I(s)/2]^2}\prod_{\mu,I}\frac{1+2\sum_{l=1}^\infty e^{-\frac{\pi^2l^2}{t}}\cosh(2\pi l[n^I_\mu(s)\pm ic^\mu_I(s)/2])}{1+2\sum_{l\in\mathbb{Z}} e^{-\frac{\pi^2l^2}{t}}}\\
=\;& W_\pm(s,m_0)\,e^{\mp i\ell_s^2\alpha(s,s_0)/2}\,e^{-\frac{t}{4}\sum_{\mu,I}[(n^I_\mu(s))^2+(c^\mu_I(s)/2)^2]}\prod_{\mu,I}\frac{1+2\sum_{l=1}^\infty e^{-\frac{\pi^2l^2}{t}}\cosh(2\pi l[n^I_\mu(s)\pm ic^\mu_I(s)/2])}{1+2\sum_{l\in\mathbb{Z}} e^{-\frac{\pi^2l^2}{t}}}
\end{aligned}
$$

where in the last step we have used $\sum_{\mu,I} n^I_\mu(s)c^\mu_I(s)=0$.

Notice that for the applications that we have in mind we have

$$n^I_\mu(s)=\sum_{l=1}^n \delta_{I,I_{m_l}}\delta^{\mu_l}_\mu \tag{6.76}$$

and

$$c^\mu_I(s)=\sum_J n^{\mu I}(s)\alpha(I,J)==\sum_{l=1}^n \eta^{\mu\mu_l}\alpha(I,I_{m_l})=\sum_{l=1}^n\eta^{\mu\mu_l}[\delta_{I,I_{m_l+1}}-\delta_{I,I_{m_l-1}}] \tag{6.77}$$

It follows that from the MD numbers $n^I_\mu(s)$ and $c^\mu_I(s)$ respectively only n respectively $2n$ are non vanishing. Thus

$$\sum_{\mu,I}[(n^I_\mu(s))^2+(c^\mu_I(s)/2)^2]=\frac{3}{2}n \tag{6.78}$$

which is actually independent of the specific configuration of $\mu_1..\mu_n$ and $I_{m_1},..,I_{m_n}$.
Next

$$\ell_s^2\alpha(s,s_0)=t\sum_{I,J}n^I_\mu(s)r^{\mu J}(s_0)\alpha(I,J)=t\sum_{l=1}^n[r^{\mu_l I_{m_l+1}}-r^{\mu_l I_{m_l+1}}] \tag{6.79}$$

We can make (6.79) vanish identically, for example, by choosing $M=|\gamma|$ to be an even number and by choosing $r^{I_m}_\mu=\sqrt{2}$ for m even and $r^{I_m}_\mu=\sqrt{3}$ for m odd. This meets all our requirements on the real numbers r^I_μ that we have imposed.
Finally, consider the product over the MD pairs (I,μ) appearing in (6.75). Precisely for the n pairs $(I_{m_l},\mu_l),\; l=1,..,n$ and for the $2n$ pairs $(I_{m_l\pm1},\mu_l),\; l=1,..,n$ the factor is different from unity. Thus (6.75) simplifies to

$$
\begin{aligned}
&\frac{<\psi^{\pm}_{\gamma,L,m_0},\pi_\omega(W_\pm(s))\psi^{\pm}_{\gamma,L,m_0}>}{||\psi^{\pm}_{\gamma,L,m_0}||^2}\\
=\;& W_\pm(s,m_0)\,e^{-\frac{3nt}{8}}\,[\frac{1+2\sum_{l=1}^\infty e^{-\frac{\pi^2l^2}{t}}\cosh(2\pi l)}{1+2\sum_{l\in\mathbb{Z}} e^{-\frac{\pi^2l^2}{t}}}]^n\,[\frac{1+2\sum_{l=1}^\infty e^{-\frac{\pi^2l^2}{t}}\cos(\pi l)}{1+2\sum_{l\in\mathbb{Z}} e^{-\frac{\pi^2l^2}{t}}}]^{2n} && (6.80)
\end{aligned}
$$

It is easy to see that the constants

$$\begin{aligned} c_1(t) &:= 2\sum_{l\in\mathbb{Z}} e^{-\frac{\pi^2 l^2}{t}} \\ c_2(t) &:= 2\sum_{l\in\mathbb{Z}} e^{-\frac{\pi^2 l^2}{t}} \cosh(2\pi l) \\ c_3(t) &:= 2\sum_{l\in\mathbb{Z}} e^{-\frac{\pi^2 l^2}{t}} \cos(\pi l) \end{aligned} \tag{6.81}$$

vanish faster than any power of t^{-1} as $t \to 0$. Hence we may finish our computation by displaying the compact formula

$$\frac{< \psi^{\pm}_{\gamma,L,m_0}, \pi_\omega(W_\pm(s))\psi^{\pm}_{\gamma,L,m_0} >}{||\psi^{\pm}_{\gamma,L,m_0}||^2} = W_\pm(s,m_0)\, e^{-\frac{3nt}{8}} \left(\frac{[1+c_2(t)][1+c_3(t)]^2}{[1+c_1(t)]^3}\right)^n \tag{6.82}$$

Obviously, the expectation value of $\pi_\omega(W_\pm(s))$ in the states $\psi^{\pm}_{\gamma,L,m_0}$, divided by the the classical value at the phase space point m_0 (a complex number of modulus one), differs from unity by a constant of order nt. We could even get exact agreement by a finite "renormalization" of the operator $\pi_\omega(W_\pm(s))$ by multiplying it by inverse of the $t-$dependent factor in (6.82) which depends on s only through the invariant quantity $n = |\gamma(s)|$ (counting non – empty intervals only).

Let us now come to the expectation value of the $\pi_\omega(Z_\pm)$. On the states $\psi^{\pm}_{\gamma,L,m_0}$ these operators reduce to

$$\begin{aligned} \pi_\omega(Z_\pm^{\mu_1..\mu_n}) &= (\frac{L}{2i})^n C_n \cdot \frac{1}{n!} \sum_{\pi\in S_n} \sum_{\sigma_1,..,\sigma_n=\pm 1} \sum_{1\le m_1\le..\le m_n\le M} \times \\ &\times \exp(\mp it/2 \sum_{j=1}^{n-1} \sum_{l=j+1}^{n} \sigma_{\pi(j)}\sigma_{\pi(l)} [n^{\pi(j)} \cdot n^{\pi(l)}][\delta_{m_{\pi(l)},m_{\pi(j)}+1} - \delta_{m_{\pi(l)},m_{\pi(j)}-1}]) \times \\ &\times \pi_\omega(W_\pm((I_{m_1}, \sigma_1 \frac{1}{L}\delta^{(1)}) \cup .. \cup (I_{m_n}, \sigma_n \frac{1}{L}\delta^{(n)})) \end{aligned} \tag{6.83}$$

where $\delta^{(l)}_\mu = \delta^{\mu_l}_\mu$. Notice that the phase in the second line of (6.83) is non-trivial only on configurations $I_{m_1}, .., I_{m_n}$ which would have zero mesure in the continuum and even in that case it differs from unity only by a term of order nt. Thus, up to nt/M corrections we can ignore that phase (alternatively we could avoid the phase right from the beginning by redifining our operators as to perform the sum over $m_{l+1} \ge m_l + 2,\ l = 1, .., n-1,\ m_n \le M,\ m_1 \ge 1$). Then taking the expectation value of (6.83) using (6.82) gives up to $O(nt, \frac{1}{M})$ corrections

$$\pi_\omega(Z_\pm^{\mu_1..\mu_n}) = L^n C_n \cdot \sum_{1\le m_1\le..\le m_n\le M} \prod_{l=1}^{n} \sin(\frac{1}{L} Y_\pm^{\mu_l}(I_{m_l}, m_0)) \tag{6.84}$$

It is clear that for sufficiently large M (6.84) is a very good approximation to $Z_\pm^{\mu_1..\mu_n}(m_0)$. In fact the limit $\lim_{M\to\infty}$ reproduces the exact integral.

This kind of calculations can be used to confirm that the terms that we claimed to be subleading in the commutation relations for the **Quantum Pohlmeyer Charges** are indeed neglible in the semiclassical limit. They can also be used to show that the relative fluctuations (absolute fluctuation divided by the square of the expectation value) of the quantum invariants are of order $O(nt/M)$. We will not display these tedious but straightforward calculations here but refer the interested reader to [36] for similar calculations performed in LQG.

6.8 Physical Hilbert Space

What we have found so far is a representation $(\pi_\omega, \mathcal{H}_\omega)$ for the algebra of quantum invariants or Dirac Observables corresponding to the classical $Z^{(n)}_\pm$. These operators commute with the unitary representations $U^\pm_\omega(\varphi)$ of the two copies of Diff(S^1) generated by the two Virasoro constraints. The $\pi_\omega(Z^{k_1..k_n}_\pm)$ transform covariantly under Poincaré transformations. What is left to do is to find the physical Hilbert space. In the present situation one can define the physical Hilbert space in two equivalent ways. The first one corresponds to gauge fixing, the second one to group averaging as defined in section 5.

Gauge Fixing:

Given any number M we fix a graph γ_M once and for all with $|\gamma_M| = M$. We then consider the gauge fixed Hilbert space $\mathcal{H}^\pm_{gf}$ defined by the completion of the finite linear span of states $\pi_\omega(W_\pm(s))\Omega_\omega$ with $\gamma(s) = \gamma_M$ whenever $|\gamma(s)| = M$. By definition, the $\pi_\omega(Z^{(n)}_\pm)$ preserve $\mathcal{H}_{gf}$. Consider any other choice $M \mapsto \gamma'_M$. For each $M = 0, 1, ..$ there is an element $\varphi \in$Diff$_\pm(S^1)$ such that $\varphi_M(\gamma_M) = \gamma'_M$. It follows that if $\gamma(s'_{kl}) = \gamma'_{M_k}$ for all $l = 1, .., L_k$ then

$$\begin{aligned}
&< [\sum_{k=1}^K \sum_{l=1}^{L_k} z_{kl}\pi_\omega(W_\pm(s'_{kl}))]\Omega_\omega, \pi_\omega(Z^{(n)}_\pm)[\sum_{k=1}^K \sum_{l=1}^{L_k} \tilde{z}_{kl}\pi_\omega(W_\pm(s'_{kl}))]\Omega_\omega >_\omega \\
= &\sum_{k=1}^K < [\sum_{l=1}^{L_k} z_{kl}\pi_\omega(W_\pm(s'_{kl}))]\Omega_\omega, \pi_\omega(Z^{(n)}_\pm)[\sum_{l=1}^{L_k} \tilde{z}_{kl}\pi_\omega(W_\pm(s'_{kl}))]\Omega_\omega >_\omega \\
= &\sum_{k=1}^K < [\sum_{l=1}^{L_k} z_{kl}\pi_\omega(W_\pm(s_{kl}))]\Omega_\omega, U^\pm_\omega(\varphi_{M_k})^{-1}\pi_\omega(Z^{(n)}_\pm)U^\pm_\omega(\varphi_{M_k})[\sum_{l=1}^{L_k} \tilde{z}_{kl}\pi_\omega(W_\pm(s_{kl}))]\Omega_\omega >_\omega \\
= &< [\sum_{k=1}^K \sum_{l=1}^{L_k} z_{kl}\pi_\omega(W_\pm(s_{kl}))]\Omega_\omega, \pi_\omega(Z^{(n)}_\pm)[\sum_{k=1}^K \sum_{l=1}^{L_k} \tilde{z}_{kl}\pi_\omega(W_\pm(s_{kl}))]\Omega_\omega >_\omega
\end{aligned} \tag{6.85}$$

so that expectation values of the invariants coincide. Here we have made use of the orthogonality relations of the states defined over different graphs. In fact, the two gauge fixed representations are unitarily equivalent because both are equivalent to direct sums of Hilbert spaces $\mathcal{H}^\pm_{\omega,\gamma_M}$ and $(\mathcal{H}^\pm_{\omega,\gamma'_M})$ respectively which are preserved by all charges and the unitary operator that maps between the Hilbert spaces is the one that maps $\mathcal{H}^\pm_{\omega,\gamma_M}$ to $\mathcal{H}^\pm_{\omega,\gamma'_M}$.

Group Averaging:

Since on the $\pi_\omega(W_\pm(s))\Omega_\omega$ the gauge group acts by diffeomorphisms, we can directly copy the analysis from [26]. We will just summarize the main results.

To each momentum network label s we assign a class $[s]$ defined by the orbit of s, that is, $[s] := \{\varphi(s);\ \varphi \in \text{Diff}(S^1)\}$. To each class $[s]$ we assign a distribution on the space Φ_{Kin}, consisting of the finite linear combinations of states of the form $\pi_\omega(W_\pm(s))\Omega_\omega$, defined by

$$\rho^\pm_{[s]}(\pi_\omega(W_\pm(s'))\Omega_\omega) := \chi_{[s]}(s') = \delta_{[s],[s']} \tag{6.86}$$

where $\chi_.$ denotes the characteristic function. These states are the images of the anti – linear rigging map

$$\rho^\pm_{[s]} := \rho(\pi_\omega(W_\pm(s))\Omega_\omega) \tag{6.87}$$

and formally we have

$$\rho^\pm_{[s]} = \sum_{s' \in [s]} < \pi_\omega(W_\pm(s'))\Omega_\omega, . >_\omega \tag{6.88}$$

which explains the word "group averaging".

The distributions $\rho_{[s]}$ belong to the dual Φ^*_{Kin} of Φ_{Kin} on which one defines duals of operators O by

$$[O'\rho^{\pm}_{[s]}](f) := \rho_{[s]}(O^\dagger f) \tag{6.89}$$

where $O^\dagger$ is the adjoint of O in $\mathcal{H}_{Kin} = \mathcal{H}_\omega$ and $f \in \Phi_{Kin}$. It follows that

$$[U^{\pm}_\omega(\varphi)]'\rho^{\pm}_{[s]} = \rho^{\pm}_{[s]} \tag{6.90}$$

is invariant, hence they solve the Virasoro constraints exactly. One defines the physical Hilbert space $\mathcal{H}^{\pm}_{Phys}$ as the completion of the finite linear span of the $\rho^{\pm}_{[s]}$ under the inner product

$$< \rho^{\pm}_{[s]}, \rho_{[s']} >_{Phys} := \rho_{[s']}(\pi_\omega(W_\pm(s))\Omega_\omega) = \delta_{[s],[s']} \tag{6.91}$$

The action of the charges $\pi_\omega(Z^{(n)}_\pm)$ is again by duality

$$[(\pi_\omega(Z^{(n)}_\pm))'\rho^{\pm}_{[s]}](f)\rho^{\pm}_{[s]}]((\pi_\omega(Z^{(n)}_\pm)f) \tag{6.92}$$

where we have used symmetry of the operators. It is not difficult to see that

$$(\pi_\omega(Z^{(n)}_\pm))'\rho(\pi_\omega(W_\pm(s))\Omega_\omega) = \rho((\pi_\omega(Z^{(n)}_\pm)\pi_\omega(W_\pm(s))\Omega_\omega) \tag{6.93}$$

so that the rigging map ρ commutes with the invariants due to diffeomorphism invariance. It follows from the general properties of a rigging map [23] that the dual operators $(\pi_\omega(Z^{(n)}_\pm))'$ are symmetric as well on $\mathcal{H}^{\pm}_{Phys}$.

Finally, it is clear that dual representation of the charges is unitarily equivalent to the gauge fixed representation above by simply identifying $\mathcal{H}^{\pm}_{Phys}$ with $\mathcal{H}^{\pm}_{gf}$.

6.9 Gravitons

It is easy to check that in our notation the graviton states in usual string theory in the lightcone gauge are given by the symmetric, transverse and traceless components of

$$|a, b; p > := [\int_{S^1} dx\; e^{ix}\; \hat{Y}^a_-]\; [\int_{S^1} dx\; e^{-ix}\; \hat{Y}^b_+]\; |p> \tag{6.94}$$

with $a, b = 1, ..D-1$ given by transversal indices and $|p>$ is the usual string theory vacuum (tachyon with momentum p). See e.g. [13] for a derivation. Due to the mode functions $e^{\pm ix}$ appearing in (6.94), graviton states are not gauge invariant states in the sense of section 6.8. We can, however, describe them in our gauge fixed Hilbert space as the massless states ($p \cdot p = 0$)

$$\Omega^{ab}_{\omega_p} := \pi_{\omega_p}(W^{aM}_-)\pi_{\omega_p}(W^{bM}_+)\Omega_{\omega_p} \tag{6.95}$$

in the limit $M \to \infty$ where (S_M denotes symmetric projection)

$$\begin{aligned} \pi_\omega(W^{aM}_\pm) &= (\frac{L}{2i})^M\; S_M \cdot \prod_{m=1}^{M} \{[\pi_\omega(W_\pm(s^a_{c,m})) - \pi_\omega(W_\pm(s^a_{c,m}))^{-1}] \mp i[\pi_\omega(W_\pm(s^a_{s,m}) - \pi_\omega(W_\pm(s^a_{s,m}))^{-1}]\} \\ s^a_{c,m} &= (\frac{\delta^a_\mu}{L}\cos(\mp\frac{2\pi i[m+1/2]}{M}), [\frac{m-1}{M}, \frac{m}{M}]), \quad s^a_{s,m} = (\frac{\delta^a_\mu}{L}\sin(\mp\frac{2\pi i[m+1/2]}{M}), [\frac{m-1}{M}, \frac{m}{M}]) \end{aligned} \tag{6.96}$$

Since, however, the operators (6.96) are not gauge invariant, it is not clear what their meaning is in the light of the invariant description of this paper. Clearly, more work is needed in order to obtain a meaningful notion of graviton creation operators in terms of the invariant charges $\pi_\omega(Z^{(n)}_\pm)$. Within LQG this has only recently been understood in the linearized sector [37] but an understanding in the full theory is still lacking. We will come back to this question in the companion paper [13].

7 Conclusions, Open Questions and Outlook

In this paper we have combined ideas from Loop Quantum Gravity, Algebraic Quantum Field Theory and Pohlmeyer's Theory of the invariant charges in order to construct quantum field theories for the closed, bosonic string in flat Minkowskian target space which differ significantly from usual string theory. Let us list once more the main differences:

1. *Target Space Dimension*
   There is no sign, neither from a ghost free spectrum requirement (covariant quantization of usual string theory) nor from a Lorentz invariance requirement (lightcone quantization of usual string theory), of a critical dimension. Our construction works in any dimension, especially $D = 4$.

2. *Ghosts*
   We always work with honest Hilbert spaces, mathematically ill – defined objects such as negative norm states are strictly avoided. Hence there are no ghosts to get rid of.

3. *Weyl Invariance*
   We never introduce a worldsheet metric because we are working directly with the more geometrical Nambu – Goto string rather than the Polyakov string. Thus, there is no artificial Weyl invariance introduced which is to be factored out later.

4. *Conformal Invariance*
   We never have to fix the (Weyl and) worldsheet diffeomorphism invariance by going to a conformal worldsheet gauge. Our formulation is manifestly worldsheet diffeomorphism invariant. Hence, there is never a residual gauge freedom corresponding to the conformal diffeomorphism group of the flat worldsheet metric to be taken care of. For the same reason, conformal field theory does not play any role whatsoever in our approach.

5. *Virasoro Anomalies and Central Charge*
   Following the tradition of algebraic QFT, we have separated the quantum algebra of string theory from its representation theory. On a properly chosen Weyl algebra of kinematical operators we have the local gauge symmetry group of the string corresponding to worldsheet diffeomorphisms and the global Poincaré symmetry acting by automorphisms. Then by standard operator theoretical constructions one obtains automatically an anomaly – free, moreover unitary representation of both symmetry groups on an important subclass of cyclic representations, provided they exist. Hence, there are no anomalies in our formulation, the central charge vanishes. This is a direct consequence of carrying out a true Dirac quantization of the constraints in contrast to standard string theory where only half of the constraints are imposed strongly. We will come back to that point in our companion paper [13]

6. *Existence of Representations*
   We did not (yet) carry out a full analysis of the representation theory of the quantum string. However, we found at least one representation which fulfills all requirements.

7. *Invariants*
   To the best of our knowledge, in standard string theory the problem of defining the classical Dirac observables as operators on the Hilbert space has not been addressed so far. The closest construction that we are aware of are the DDF operators in the lightcone gauge of the Virasoro constraints [1]. In a fully worldsheet background independent and diffeomorphism invariant formulation that one is forced to from an LQG perspective, dealing with the Dirac observables is mandatory. Fortunately, the invariant charges have been constructed already by Pohlmeyer and his collaborators. The example representation that we have constructed actually supports a specific quantum deformation of the classical charge algebra. The corresponding operators define invariant n – point functions which are finite without UV – divergences.

8. *Tachyon*
   We saw that we can construct representations with arbitrary, non – negative mass spectrum , so there is no tachyon in the spectrum. Usually the tachyon is (besides the phenomenological need for fermionic matter) one of the motivations for considering the superstring. Our example shows that this depends on the representation and is not always necessary.

In the jargon of standard string theory, one could summarize this by saying that *the LQG – String presents a new and consistent solution to quantizing string theory.* Actually, there is not *the* LQG – String, presumably there exist infinitely many solutions to the representation problem (which are consistent by definition).

Of course, we do not claim that the particular representation we found is necessarily of any physical significance. In fact it cannot be since we have not included (yet) any fermionic degrees of freedom. Also, besides not having carried out a full analysis of the representation theory, our analysis is incomplete in many respects as for instance we have not yet developed the S – Matrix theory for the LQG – String (however, sinze the **Pohlmeyer Charges** are nothing else than invariant n – point functions, this is presumably not very difficult). What is also missing, so far, is a comparison with the objects of usual string theory because it is hard to translate gauge dependent notions such as graviton states into our invariant language, see section 6.9. Finally, there are four immediate open questions:
i)
First of all the **Pohlmeyer Charges** together with the the boost generators reconstruct the string embedding X^μ (up to gauge transformations) completely only up to scalar multiples of the momentum p^μ. It seems to be hard to define an invariant which captures this one parameter degree of freedom unless one includes string scattering [7, 20].
ii)
Secondly, an interesting open question is whether one can find a supersymmetric (or at least fermionic) extension of our Weyl algebra and if curved target spaces can be treated the same way. What is needed is an analogue of the $Y_\pm$ with the same simple commutation relations and the same simple behaviour under gauge transformations. If that would be possible and if appropriately generalized **Pohlmeyer Charges** could be found, then one could repeat the analysis of this paper because the structure of the constraint algebra of the $V_\pm$ remains the same even for the supersymmetric extension and for curved target spaces.
iii)
Thirdly one might wonder whether an approach based on invariants as carried out in this paper is not possible also for higher p – brane theories such as the (super)membrane [12] which is a candidate for M – Theory.
iv)
Lastly one may wonder which other GNS representations one gets by constructing the folium of ω. Of course, the folium should be based on $G-$invariant positive trace class operators, see section 4. Thus one would try to define those from bounded operators constructed from the $\pi_\omega(Z_\pm^{(n)})$. Notice that trace class operators are in particular compact, thus they must have discrete spectrum with all non vanishing eigenvalues of finite multiplicity. Since $\mathcal{H}_\omega$ is not separable, such an operator would have to have uncountably infinite multiplicity for the eigenvalue zero.

Let us conclude by stressing once more that the claim of this paper is certainly not to have found a full solution of string theory. Rather, we wanted to point out two things:
First of all, that canonical and algebraic methods can be fruitfully combined in order to analyze the string. Secondly, that the specific Fock representation that one always uses in string theory is by far not the end of the story: The invariant representation theory of the quantum string, as we have defined it here, is presumably very rich and we encourage string theorists to study the string from the algebraic perspective and to systematically analyze all its representations. This might lead to a natural resolution of major current puzzles in string theory, such as the cosmological constant puzzle [38] (120 orders of magnitude too large), the

tachyon condensation puzzle [39] (unstable bosonic string vacua), the vacuum degeneracy puzzle [40] (huge moduli space of vacua upon compactification), the phenomeology puzzle [41] (so far the standard model has not been found among all possible string vacua of the five superstring theories currently defined, even when including D – branes) and finally the puzzle of proving perturbative finiteness beyond two loops [42]. See the beautiful review [43] for a status report on these issues. Namely, it might be that there are much simpler representations of the string, especially in lower dimensions and possibly without supersymmetry, which avoid or simplify all or some these problems.

While this would be attractive, the existence of new, phenomenologically sensible representations would demonstrate that $D = 10, 11, 26$ dimensions, supersymmetry and the matter content of the world are tied to a specific representation of string theory and hence would not be a prediction in this sense. We believe, however, that the potential discovery of new, physically interesting representations for string theory, in the sense of this paper, is a fascinating research project which could lead to major progress on the afore mentioned puzzles.

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